On One Scheme of Singular Integrals Approximate Calculation and on Its Use

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Abstract   The article deals with construction of one specific scheme for approximate calculation of singular integrals and its use for numerical solutions of integral equations of some class in the sense of Cauchy principal value. Composed quadrature formulas makes it possible equal assessment for quite a wide variety of smooth closed contours.

Keywords   Integral, Continuity model, Liapunov curve, Singular integral equation

1. Introduction

Numerous problems of mathematics, physics, and mechanics are reduced to singular integral equations [1] that predetermines much attention to the issues of numerical solution of the mentioned equations. In its turn, it is natural that the one or another approximate solution scheme for such equation may be constructed based exactly on the approximate calculation schemes of singular integrals with a Cauchy kernel. Thus, in addition to the fact that the mentioned integrals represent in themselves the interesting and attractive mathematical object, the development of the approximate solution methods for equations consisting of such integrals is of considerable interest, as well.

The following main notation is used in the work: L is the oriented contour on the plane; C(L) denotes the class of functions continuous on L with a norm $\|\varphi\|_c = \max_{t \in L} |\varphi(t)|$; $H_\alpha(L)$ denotes the class of Hölder functions on L with an exponent α(0<α≤1) and a norm $\|\varphi\|_{H_\alpha} = \max_{t \in L} |\varphi(t)| + \sup_{t_1,t_2 \in L} \frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^\alpha}$. besides,

$$S(\varphi; t_0) = \frac{1}{\pi i} \int_L \frac{\varphi(t)}{t - t_0} dt \ (t_0 \in L) \quad (1)$$

2. Main Body

Let us assume that $t = t(s) = x(s) + iy(s)$ is a parametric equation of some, sufficiently smooth, closed L contour referred to arc abscisse $\left(0 \le s < l, h = \frac{l}{n}\right)$. We select two $\{\tau_\sigma\}_{\sigma=0}^{n-1}$ and $\{\tilde{\tau}_\vartheta\}_{\vartheta=0}^{n-1}$ systems of nodes on L as follows: $\tau_\sigma = t(s_\sigma) \ (s_\sigma = \frac{l}{n} \sigma = h\sigma)$ divides L into n

equal parts (according to arc length), while $\tilde{\tau}_\vartheta$ point is a midpoint of $\tau_\vartheta \ \tau_{\vartheta+1}$ arc. We assume that $\tau_n = \tau_0, \ \tau_{n-1} = \tau_{-1}$.

Denote

$$S_n(\varphi; \tilde{\tau}_\vartheta) = \sum_{\sigma=0}^{n-1} P_\sigma \frac{\varphi(\tau_\sigma)}{\tau_\sigma - \tilde{\tau}_\vartheta} \quad (2)$$

where $P_0 = P_0^* + P_{n-1}^*$, $P_\sigma = P_{\sigma-1}^* + P_\sigma^*$ $(\sigma = 1,2,\ldots,n-1)$ and $P_j^* = \frac{\tau_{j+1} - \tau_j}{2\pi i}$ (j=0,1, . . . ,n-1).

The following theorem is proved:

if $\varphi(t) \in H_\alpha(L)(0 < \alpha \le 1)$, then the following estimation takes place

$$\max_{0 \le \tau \le n-1} |S(\varphi; \tilde{\tau}_\vartheta) - S_n(\varphi; \tilde{\tau}_\vartheta)| \quad (3)$$
$$= O(\sqrt{\omega(h_n)} \ln j_n + j_n^{-1} + n^{-\alpha} \ln n)$$

where ω denotes the biggest among modules of continuity of functions $x'(s), y'(s) \ (0 \le s \le e)$, j_n is some sequence of natural numbers, satisfying the conditions $j_n \to \infty$ and $\frac{j_n}{n} \to 0 (n \to \infty)$. For any natural $\vartheta(0 \le \vartheta \le n - 1) \ h_n$ denotes the length of that $l_{n\vartheta}$ arc, which has endpoints $\tau_{\vartheta - \left[\frac{j_n - 2}{2}\right]}$ and $\tau_{\vartheta + 1 + \left[\frac{j_n - 2}{2}\right]}$, midpoint $\tilde{\tau}_\vartheta$, consists of exactly j_n nodes and $h_n \to 0$, when $n \to \infty$ so that $\frac{h}{h_n} \to 0 (n \to \infty)$. At that, the constant entering the right side of (3) depends on Hölder constant of function φ.

The mentioned estimation is fair for any smooth, closed contour. In case, when $\sqrt{\omega(h_n)} \ln j_n \to 0 (n \to \infty)$, the process is uniformly convergent for any $\vartheta(0 \le \vartheta \le n - 1$. If we assume that $j_n = [n^\mu](0 < \mu < 1)$ and L is a Liapunov curve with some exponent $\lambda(0 < \lambda \le 1$, the estimation (3) converts into the following $\max_{0 \le \tau \le n-1} |S(\varphi; \tilde{\tau}_\vartheta) - S_n(\varphi; \tilde{\tau}_\vartheta)| = O(n^{\frac{\lambda\mu - \lambda}{2}} \ln n + n^{-\mu} + n^{-\alpha} \ln n)$.

The considered scheme may be used for numerical solution of singular integral equations of I kind, in particular for the following equation

$$\frac{1}{\pi i} \int_L \frac{\varphi(t)}{t-t_0} dt + \frac{1}{\pi i} \int_L K(t_0,t)\varphi(t)\, dt = f(t_0), \quad t_0 \in L \quad (4)$$

Here L is a closed, smooth Liapunov curve, while $K(t_0,t)$ and $f(t_0)$ are Hölder class functions given on L. We associate (4) with the system of linear algebraic equations

$$\sum_{\vartheta=0}^{n-1} \frac{P_\sigma}{\tau_\sigma - \tilde{\tau}_\vartheta} \varphi_n(\tau_\sigma) + \sum_{\vartheta=0}^{n-1} P_\sigma K(\tilde{\tau}_\vartheta, \tau_\sigma) \varphi_n(\tau_\sigma)$$

$$= f(\tilde{\tau}_\vartheta) \ (\vartheta = 0,1,\dots,n-1) \qquad (5)$$

It is proved that starting with certain n, the system (5) is unambiguously solvable and values $\varphi_n(\tau_\sigma)(\sigma = 0,\dots,\lambda-1)$ uniformly converge to the solution corresponding with (4).

3. Conclusions

There are composed quadrature formulas, which enable us to give an equal assessment for quite wide class of smooth closed L contours, in contradistinction from the cases of similar schemes considered by other authors [2], [3], when an integrable curve is either a segment or a circle.

As for the scheme application in approximate solutions of singular integral equations, the following circumstance should be noted: a considered approximate scheme, which can lay a foundation for approximation of singular equation with Cauchy kernel, according to known terminology, is ranked among direct calculation schemes, use of which in practice for numerical solutions is much easier, than of those schemes, which are based on equivalent regularization of singular equations (reduction to Fredholm equation).

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