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Process Location Control Chart based on Gastwirth Estimator

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Abstract Control charts are widely used in quality control to monitor and detect changes in the process location and variability. In this paper, we present a control chart based on Gastwirth estimator for monitoring the process location. The performance of the proposed control chart is evaluated through various performance measures providing a comprehensive analysis of its shift detection capability. The proposed control chart is compared with its competitors. The control chart is demonstrated through an example to showcase its application in reality.

Keywords Control limits, Gastwirth estimator, Median run length, Order statistic, Outliers, Process location

1. Introduction

Control charts are vital tools in statistical quality control, offering a visual means of monitoring process parameters over time. They enable practitioners to distinguish between chance and assignable causes of variation, which signals potential issues that require intervention. By continuously monitoring process mean and variance, control charts facilitate timely decision-making and help maintain quality products. Specifically, control charts for monitoring process location are crucial for ensuring that the mean of a process remains stable over time. The early work of Shewhart (1931) laid the foundation for control charts which assume normality and utilize 3-sigma control limits to detect shifts in process mean. Page (1961) introduced cumulative sum control charts (CUSUM) which are sensitive to detect small shifts in the process mean. Roberts (1959) proposed exponentially weighted moving average (EWMA) control chart which assigns weights to previous observations, making it more sensitive to detect small shifts. Montgomery (1996) provides a comprehensive overview of control charts, discussing various types of control charts, their underlying concepts and applications in monitoring process stability in detecting shifts in process parameters.

Many control charts for process location parameter rely on the assumption that the underlying process has normal distribution. However, in practice, this assumption need not be valid. For example, in glass industry, the thickness of glass sheets used for windows, screens or optical applications is critical to monitor the process parameter. The thickness measurements are often symmetrically distributed around a target value due to production techniques like the float glass process. Some slight variations in the cooling process or other external factors can result in occasional extreme deviations which lead to a heavy-tailed distribution, where control charts based on normality may fail to detect these changes.

As a result, it is essential to propose control charts that do not rely on the assumption of normal distribution. To address this issue, it is necessary to develop control charts that are resilient to non normality. Some significant studies that do not rely on the assumption of normality include control charts based on sign statistics proposed by Amin et al. (1995), distribution-free Shewhart type control chart based on signed ranks developed by Bakir (2004), synthetic control chart developed by Pawar and Shirke (2010), Shewhart type nonparametric control chart based on the run statistic suggested by Zombade and Ghute (2019). A comprehensive overview of nonparametric control charts, detailing various methodologies for quality control is studied by Chakraborti (2019). Here, the author emphasizes the robustness and applicability of nonparametric approaches in statistical process control providing practical insights and examples for practitioners. Bhat and Patil (2024a and 2024b) propose control charts for process location based on sample median and sample midrange respectively to monitor the process location parameter, employing Shewhart's methodology.

In this paper, we propose a control chart that employs a robust statistical measure to improve monitoring accuracy and enhance shift detection in process location, especially in the presence of outliers and under non normal distributions. Section 2 deals with proposed control chart and section 3 discusses the evaluation of proposed control chart. In sections 4 and 5, we respectively deal with illustration of the

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proposed control chart through examples and conclusions based on our observations.

2. Control Chart Based on Gastwirth Estimator

In this section, we propose control chart based on Gastwirth estimator (G) due to Gastwirth (1966) which is a robust estimator. Suppose $X_1, X_2, \dots, X_n$ is a random sample of size n from a distribution with cumulative density function $F(x)$, then G is given by

$$G = 0.3Q_{\frac{1}{3}}(x) + 0.4Q_{\frac{1}{2}}(x) + 0.3Q_{\frac{2}{3}}(x) \quad (1)$$

where Q_p is p^{th} sample quantile.

Also, suppose $X_{(i)}$ represents i^{th} order statistic, then

$$G = 0.3X_{\left(\frac{n}{3}\right)} + 0.4X_{\left(\frac{n}{2}\right)} + 0.3X_{\left(\frac{2n}{3}\right)} \quad (2)$$

where $X_{\left(\frac{n}{3}\right)}$, $X_{\left(\frac{n}{2}\right)}$ and $X_{\left(\frac{2n}{3}\right)}$ represent 33rd, 50th and 66th percentiles of the data. The estimator is useful from practical point of view as it has better efficiency than median and is also robust. It has better computational efficiency than other robust estimators like estimators due to Harrel and Davis (1982) and Hodges and Lehmann (1963). Also, it is helpful in estimating the location parameter in the presence of outliers or when process has non-normal distribution. Its approach of using fixed weights enable its utility across various symmetric distributions.

According to DasGupta (2008), G has asymptotically normal distribution with mean

$$\mu(G) = 0.3\xi_{\frac{1}{3}} + 0.4\xi_{\frac{1}{2}} + 0.3\xi_{\frac{2}{3}} \quad (3)$$

and variance σ_G^2, where $\xi_{\frac{1}{3}}$, $\xi_{\frac{1}{2}}$ and $\xi_{\frac{2}{3}}$ are respectively the population quantiles corresponding to the 33rd, 50th and 66th percentiles.

$$\text{That is, } G \sim N(\mu(G), \sigma_G^2) \quad (4)$$

where

$$\sigma_G^2 = \frac{9}{100}\left(\frac{p_1 q_1}{nf^2(\xi_{p_1})}\right) + \frac{16}{100}\left(\frac{p_2 q_2}{nf^2(\xi_{p_2})}\right) + \frac{9}{100}\left(\frac{p_3 q_3}{nf^2(\xi_{p_3})}\right) + \frac{24}{100}\left(\frac{p_1 q_2}{nf(\xi_{p_1})f(\xi_{p_2})}\right) + \frac{18}{100}\left(\frac{p_1 q_3}{nf(\xi_{p_1})f(\xi_{p_3})}\right) + \frac{24}{100}\left(\frac{p_2 q_3}{nf(\xi_{p_2})f(\xi_{p_3})}\right),$$

for $p_1 < p_2 < p_3$ with $p_1 = \frac{1}{3}$, $p_2 = \frac{1}{2}$, $p_3 = \frac{2}{3}$ and $q_j = 1 - p_j$ for $j = 1, 2, 3$.

Here, $\xi_r = F^{-1}(r)$ and $f(\xi_r)$ is value of density function at the specified population quantile ξ_r.

Therefore,

$$\sigma_G^2 = \frac{9}{100}\left(\frac{2}{9nf^2\left(\xi_{\frac{1}{3}}\right)}\right) + \frac{16}{100}\left(\frac{1}{4nf^2\left(\xi_{\frac{1}{2}}\right)}\right) + \frac{9}{100}\left(\frac{2}{9nf^2\left(\xi_{\frac{2}{3}}\right)}\right) + \frac{24}{100}\left(\frac{1}{6nf\left(\xi_{\frac{1}{3}}\right)f\left(\xi_{\frac{1}{2}}\right)}\right) + \frac{18}{100}\left(\frac{1}{9nf\left(\xi_{\frac{1}{3}}\right)f\left(\xi_{\frac{2}{3}}\right)}\right) + \frac{24}{100}\left(\frac{1}{6nf\left(\xi_{\frac{1}{2}}\right)f\left(\xi_{\frac{2}{3}}\right)}\right).$$

$$= \frac{1}{50n}\left(\frac{1}{f^2\left(\xi_{\frac{1}{3}}\right)} + \frac{2}{f^2\left(\xi_{\frac{1}{2}}\right)} + \frac{1}{f^2\left(\xi_{\frac{2}{3}}\right)}\right) + \frac{1}{25n}\left(\frac{1}{f\left(\xi_{\frac{1}{3}}\right)f\left(\xi_{\frac{1}{2}}\right)} + \frac{1}{2f\left(\xi_{\frac{1}{3}}\right)f\left(\xi_{\frac{2}{3}}\right)} + \frac{1}{f\left(\xi_{\frac{1}{2}}\right)f\left(\xi_{\frac{2}{3}}\right)}\right). \quad (5)$$

The variance expression captures the estimator's variability as a function of the sample size n and the values of the density function f (.) at the specified population quantiles $\xi_{\frac{1}{3}}$, $\xi_{\frac{1}{2}}$ and $\xi_{\frac{2}{3}}$.

Under the assumption that, a random sample of size n is taken from a symmetric or skewed distribution with location parameter (l) and scale parameter (b), following Shewhart (1931), we propose G control chart

$$UCL_G = \mu(G) + 3\sigma_G, \quad (6)$$

$$CL_G = \mu(G) \quad (7)$$

$$\text{and } LCL_G = \mu(G) - 3\sigma_G \quad (8)$$

where UCL_G, CL_G and LCL_G are respectively upper control limit, centre line and lower control limit of G control chart. And σ_G is standard deviation (sd) of G.

3. Performance of G Control Chart

In this section, we evaluate the performance of the G control chart for various distributions under consideration. To assess the performance of the proposed G control chart, we compute mean and sd of G control chart under uniform (U), exponential (E), normal (N), logistic (LG), Laplace (L) and Cauchy (C) distributions. The distributions are modified in terms of l and b such that they have mean μ and variance λ^2. As G approximately follows normal distribution with mean $\mu(G)$ and $sd\ \sigma_G$ given by square root of (5), we have to evaluate $\xi_{\frac{1}{3}}$, $\xi_{\frac{1}{2}}$ and $\xi_{\frac{2}{3}}$ under various distributions. For example, to evaluate $\xi_{\frac{1}{3}}$ under exponential distribution,

$$f(x) = \frac{1}{\lambda}e^{-\left(\frac{x-\mu+\lambda}{\lambda}\right)},\ \lambda > 0,$$
$$-\infty < \mu < \infty,\ x \geq (\mu - \lambda), \quad (9)$$

we have $F\left(\xi_{\frac{1}{3}}\right) = \frac{1}{3}$

$$\Rightarrow \int_{(\mu-\lambda)}^{\xi_{\frac{1}{3}}} f(x)\, d(x) = \frac{1}{3}$$

$$\Rightarrow \int_{\mu-\lambda}^{\xi_{\frac{1}{3}}} \frac{1}{\lambda}e^{-\left(\frac{x-\mu+\lambda}{\lambda}\right)} d(x) = \frac{1}{3}$$

putting $y = x - \mu + \lambda$

$$\Rightarrow \int_0^{\xi_{\frac{1}{3}}-\mu+\lambda} \frac{1}{\lambda}e^{-\frac{y}{\lambda}} d(y) = \frac{1}{3}$$

$$\Rightarrow \left[1-e^{-\left(\frac{\xi_{\frac{1}{3}}-\mu+\lambda}{\lambda}\right)}\right]=\frac{1}{3}$$

$$\Rightarrow \xi_{\frac{1}{3}}-\mu+\lambda=-\lambda log\left(1-\frac{1}{3}\right). \qquad (10)$$

Therefore, $\xi_{\frac{1}{3}}=\mu-\lambda\left(1+\log\left(1-\frac{1}{3}\right)\right)$. (11)

Substituting (11) in (9), we get

$$f\left(\xi_{\frac{1}{3}}\right)=\frac{1}{\lambda}e^{-\left(\frac{\mu-\lambda\left(1+\log\left(1-\frac{1}{3}\right)\right)-\mu+\lambda}{\lambda}\right)},$$

$$=\frac{1}{\lambda}e^{-\left(\frac{\lambda\left(-1-\log\left(1-\frac{1}{3}\right)+1\right)}{\lambda}\right)}$$

$$=\frac{2}{3\lambda}. \qquad (12)$$

On similar lines, other values of $f(.)$ in (5) are evaluated and σ_G is obtained. The sd and width (w_G) of G control chart for the distributions under consideration is furnished in Table 1.

The sd of the G estimator varies across distributions based on their differences in spread and tail behavior. The G statistic under Cauchy distribution has highest sd and under Laplace distribution has lowest sd. The width of G control chart is highest for Cauchy distribution followed by uniform distribution and is lowest for Laplace distribution followed by exponential distribution.

To evaluate the performance of G control chart, we consider various performance measures such as power (P_G), average run length (ARL_G), median run length (MRL_G) and standard deviation of run length ($SDRL_G$). These measures provide a comprehensive view of the control chart's ability to detect shifts in the process location. The P_G refers to the probability, ARL_G represents the expected number of samples required and MRL_G gives the median number of samples needed to detect a shift in process location. A smaller ARL_G and MRL_G value indicates quicker detection of shifts. Additionally, the $SDRL_G$ evaluates the consistency of shift detection, where a lower value indicates better consistency. Various measures are given by

$$P_G=1-\beta_G, \qquad (13)$$

where $\beta_G=P(LCL_G<G<UCL_G|\mu'=\mu+a)$, (14)

$$ARL_G=\frac{1}{P_G}, \qquad (15)$$

$$MRL_G=\frac{\log(0.5)}{\log(1-P_G)} \qquad (16)$$

and $SDRL_G=\sqrt{ARL_G(1-ARL_G)}$. (17)

The values of P_G, ARL_G, MRL_G and $SDRL_G$ for uniform, exponential, normal, logistic and Laplace distributions are determined by setting $\lambda^2=1$ and for Cauchy distribution by setting $\lambda=0.2605$ due to Arnold (1965). For sample sizes $n=5,10,15,20$ and various positive shifts, the computed values of P_G, ARL_G are provided in Table 2 and MRL_G, $SDRL_G$ are provided in Table 3. ARL_G and MRL_G of G control chart are plotted in Figure 1 for $n=10$.

Table 1. σ_G and w_G of G control chart for various distribtions

Distribution	Uniform	Exponential	Normal	Logistic	Laplace	Cauchy
σ_G	$1.4696\frac{\lambda}{\sqrt{n}}$	$0.9137\frac{\lambda}{\sqrt{n}}$	$1.1210\frac{\lambda}{\sqrt{n}}$	$1.0007\frac{\lambda}{\sqrt{n}}$	$0.7681\frac{\lambda}{\sqrt{n}}$	$1.5812\frac{\lambda}{\sqrt{n}}$
w_G	$8.8181\frac{\lambda}{\sqrt{n}}$	$5.4826\frac{\lambda}{\sqrt{n}}$	$6.7260\frac{\lambda}{\sqrt{n}}$	$6.0042\frac{\lambda}{\sqrt{n}}$	$4.6086\frac{\lambda}{\sqrt{n}}$	$9.4872\frac{\lambda}{\sqrt{n}}$

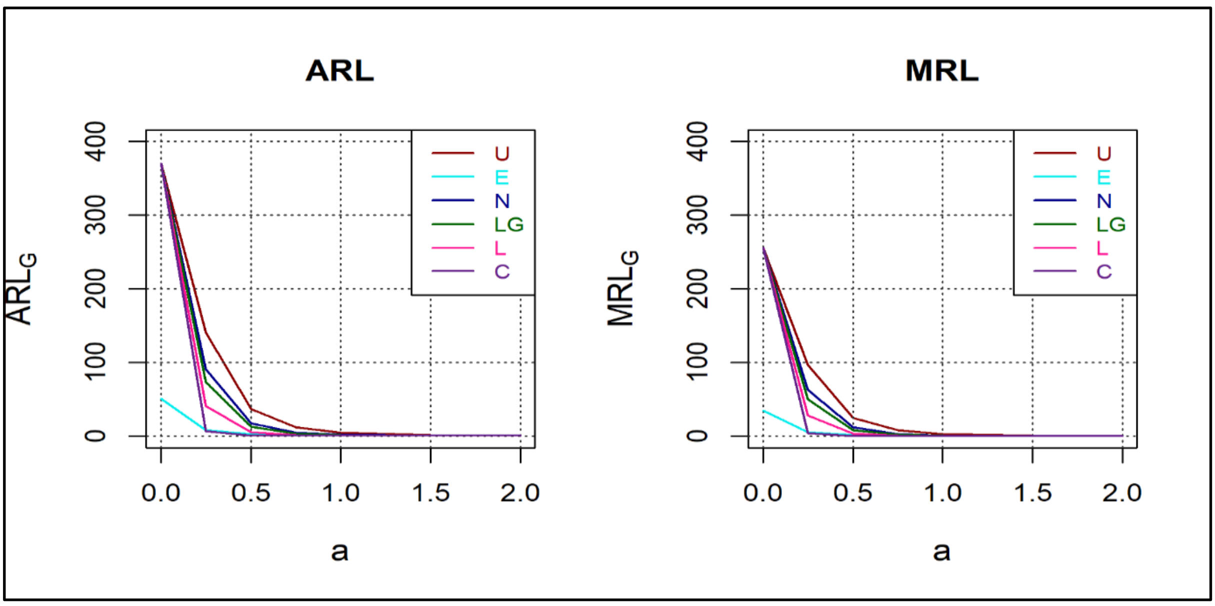

Figure 1. ARL_G and MRL_G of G Control Chart

Table 2. $\boldsymbol{P_G}$ and $\boldsymbol{ARL_G}$ for various distributions

Distribution		P_G				ARL_G			
	a \ n	5	10	15	20	5	10	15	20
Uniform	0.00	0.0027	0.0027	0.0027	0.0027	370.3983	370.3983	370.3983	370.3983
	0.25	0.0048	0.0071	0.0097	0.0127	209.9464	140.6819	102.6923	79.0280
	0.50	0.0127	0.0272	0.0463	0.0696	79.0280	36.7789	21.6193	14.3609
	0.75	0.0315	0.0828	0.1530	0.2364	31.7086	12.0720	6.5352	4.2295
	1.00	0.0696	0.1981	0.3576	0.5171	14.3609	5.0474	2.7961	1.9338
	1.50	0.2364	0.5900	0.8297	0.9411	4.2295	1.6950	1.2053	1.0625
	2.00	0.5171	0.9038	0.9884	0.9990	1.9338	1.1065	1.0117	1.0010
Exponential	0.00	0.0099	0.0197	0.0322	0.0473	101.2281	50.7160	31.0340	21.1245
	0.25	0.0424	0.1160	0.2148	0.3272	23.5998	8.6219	4.6545	3.0567
	0.50	0.1330	0.3706	0.6064	0.7810	7.5164	2.6980	1.6491	1.2803
	0.75	0.3084	0.7037	0.9082	0.9772	3.2425	1.4211	1.1011	1.0233
	1.00	0.5443	0.9193	0.9916	0.9994	1.8371	1.0878	1.0085	1.0006
	1.50	0.9090	0.9991	1.0000	1.0000	1.1001	1.0009	1.0000	1.0000
	2.00	0.9947	1.0000	1.0000	1.0000	1.0053	1.0000	1.0000	1.0000
Normal	0.00	0.0027	0.0027	0.0027	0.0027	370.3983	370.3983	370.3983	370.3983
	0.25	0.0064	0.0110	0.0164	0.0226	155.7527	91.0809	61.0304	44.1685
	0.50	0.0226	0.0560	0.1016	0.1574	44.1685	17.8641	9.8427	6.3538
	0.75	0.0663	0.1883	0.3414	0.4969	15.0827	5.3113	2.9295	2.0127
	1.00	0.1574	0.4290	0.6754	0.8388	6.3538	2.3312	1.4805	1.1922
	1.50	0.4969	0.8909	0.9855	0.9986	2.0127	1.1224	1.0148	1.0014
	2.00	0.8388	0.9959	1.0000	1.0000	1.1922	1.0041	1.0000	1.0000
Logistic	0.00	0.0027	0.0027	0.0027	0.0027	370.3983	370.3983	370.3983	370.3983
	0.25	0.0075	0.0136	0.0211	0.0299	133.3107	73.3875	47.4216	33.4667
	0.50	0.0299	0.0778	0.1434	0.2219	33.4667	12.8547	6.9718	4.5058
	0.75	0.0927	0.2643	0.4612	0.6374	10.7863	3.7836	2.1685	1.5689
	1.00	0.2219	0.5635	0.8078	0.9290	4.5058	1.7747	1.2379	1.0764
	1.50	0.6374	0.9590	0.9975	0.9999	1.5689	1.0427	1.0025	1.0001
	2.00	0.9290	0.9995	1.0000	1.0000	1.0764	1.0005	1.0000	1.0000
Laplace	0.00	0.0027	0.0027	0.0027	0.0027	370.3983	370.3983	370.3983	370.3983
	0.25	0.0116	0.0244	0.0410	0.0612	85.9613	40.9781	24.3974	16.3280
	0.50	0.0612	0.1732	0.3160	0.4646	16.3280	5.7731	3.1645	2.1525
	0.75	0.2071	0.5349	0.7828	0.9141	4.8295	1.8694	1.2775	1.0939
	1.00	0.4646	0.8680	0.9794	0.9976	2.1525	1.1521	1.0210	1.0024
	1.50	0.9141	0.9993	1.0000	1.0000	1.0939	1.0007	1.0000	1.0000
	2.00	0.9976	1.0000	1.0000	1.0000	1.0024	1.0000	1.0000	1.0000
Cauchy	0.00	0.0027	0.0027	0.0027	0.0027	370.3983	370.3983	370.3983	370.3983
	0.25	0.0502	0.1399	0.2580	0.3875	19.9158	7.1475	3.8753	2.5804
	0.50	0.3875	0.7991	0.9556	0.9924	2.5804	1.2514	1.0465	1.0076
	0.75	0.8580	0.9971	1.0000	1.0000	1.1655	1.0029	1.0000	1.0000
	1.00	0.9924	1.0000	1.0000	1.0000	1.0076	1.0000	1.0000	1.0000
	1.50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	2.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3. $\boldsymbol{MRL_G}$ and $\boldsymbol{SDRL_G}$ for various distributions

Distribution	a \ n	MRL_G 5	10	15	20	$SDRL_G$ 5	10	15	20
Uniform	0.00	256.3938	256.3938	256.3938	256.3938	369.8980	369.8980	369.8980	369.8980
	0.25	145.1769	97.1663	70.8337	54.4307	209.4458	140.1810	102.1910	78.5264
	0.50	54.4307	25.1450	14.6360	9.6035	78.5264	36.2755	21.1133	13.8519
	0.75	21.6303	8.0161	4.1737	2.5695	31.2046	11.5612	6.0144	3.6959
	1.00	9.6035	3.1392	1.5660	0.9522	13.8519	4.5198	2.2410	1.3438
	1.50	2.5695	0.7775	0.3916	0.2447	3.6959	1.0853	0.4974	0.2578
	2.00	0.9522	0.2961	0.1555	0.1006	1.3438	0.3432	0.1089	0.0319
Exponential	0.00	69.8188	34.8060	21.1627	14.2930	100.7269	50.2135	30.5299	20.6185
	0.25	16.0091	5.6225	2.8657	1.7493	23.0944	8.1065	4.1243	2.5073
	0.50	4.8552	1.4969	0.7434	0.4563	6.9986	2.1404	1.0347	0.5991
	0.75	1.8797	0.5699	0.2903	0.1833	2.6965	0.7736	0.3337	0.1545
	1.00	0.8818	0.2754	0.1452	0.0941	1.2401	0.3091	0.0927	0.0252
	1.50	0.2891	0.0984	0.0549	0.0370	0.3318	0.0296	0.0018	0.0001
	2.00	0.1321	0.0483	0.0280	0.0193	0.0729	0.0008	0.0000	0.0000
Normal	0.00	256.3938	256.3938	256.3938	256.3938	369.8980	369.8980	369.8980	369.8980
	0.25	107.6126	62.7853	41.9555	30.2674	155.2519	90.5796	60.5283	43.6657
	0.50	30.2674	12.0325	6.4697	4.0477	43.6657	17.3569	9.3293	5.8324
	0.75	10.1040	3.3229	1.6599	1.0091	14.5741	4.7852	2.3775	1.4276
	1.00	4.0477	1.2371	0.6160	0.3798	5.8324	1.7616	0.8434	0.4787
	1.50	1.0091	0.3128	0.1638	0.1057	1.4276	0.3707	0.1223	0.0378
	2.00	0.3798	0.1262	0.0694	0.0463	0.4787	0.0645	0.0068	0.0006
Logistic	0.00	256.3938	256.3938	256.3938	256.3938	369.8980	369.8980	369.8980	369.8980
	0.25	92.0569	50.5209	32.5224	22.8491	132.8097	72.8858	46.9190	32.9630
	0.50	22.8491	8.5589	4.4770	2.7621	32.9630	12.3446	6.4525	3.9745
	0.75	7.1243	2.2583	1.1210	0.6833	10.2741	3.2453	1.5918	0.9447
	1.00	2.7621	0.8362	0.4202	0.2620	3.9745	1.1725	0.5426	0.2867
	1.50	0.6833	0.2169	0.1158	0.0758	0.9447	0.2110	0.0503	0.0103
	2.00	0.2620	0.0900	0.0504	0.0341	0.2867	0.0212	0.0010	0.0000
Laplace	0.00	256.3938	256.3938	256.3938	256.3938	369.8980	369.8980	369.8980	369.8980
	0.25	59.2366	28.0558	16.5620	10.9675	85.4598	40.4750	23.8922	15.8201
	0.50	10.9675	3.6441	1.8250	1.1095	15.8201	5.2494	2.6172	1.5750
	0.75	2.9876	0.9054	0.4540	0.2823	4.3006	1.2748	0.5954	0.3206
	1.00	1.1095	0.3423	0.1785	0.1148	1.5750	0.4186	0.1464	0.0489
	1.50	0.2823	0.0963	0.0538	0.0362	0.3206	0.0274	0.0016	0.0001
	2.00	0.1148	0.0425	0.0248	0.0171	0.0489	0.0003	0.0000	0.0000
Cauchy	0.00	256.3938	256.3938	256.3938	256.3938	369.8980	369.8980	369.8980	369.8980
	0.25	13.4550	4.5990	2.3224	1.4138	19.4093	6.6287	3.3380	2.0194
	0.50	1.4138	0.4318	0.2226	0.1420	2.0194	0.5608	0.2206	0.0877
	0.75	0.3551	0.1187	0.0655	0.0438	0.4392	0.0541	0.0050	0.0004
	1.00	0.1420	0.0516	0.0298	0.0205	0.0877	0.0012	0.0000	0.0000
	1.50	0.0438	0.0176	0.0107	0.0076	0.0004	0.0000	0.0000	0.0000
	2.00	0.0205	0.0087	0.0054	0.0039	0.0000	0.0000	0.0000	0.0000

From Table 2 and 3, for all values of n, we observe that, P_G is 0.0027 for symmetric distributions under consideration at zero shift. However, since $E(G) = \mu - 0.2714\lambda$ for exponential distribution, P_G is 0.0027 at $a = -0.27$. The

P_G increases as a and n increase. For fixed n and small a, P_G is highest for Cauchy distribution and lowest for uniform distribution. We notice that, under symmetric distributions, ARL_G and MRL_G are approximately 370 and 256 respectively for zero shift. Under exponential distribution, these values are achieved at $a = -0.27$. Also, ARL_G and MRL_G being smaller, vary for different n for zero shift. For fixed n, as a increases, ARL_G and MRL_G decrease, exhibiting effective shift detecting capability of the control chart. The same nature in ARL_G and MRL_G is observed for fixed a and increasing n. Also, $SDRL_G$ decreases for increasing a and n, indicating stability of the control chart in detecting the shifts.

From Figure 1, it is observed that, among symmetric distributions, for specified value of a, both ARL_G and MRL_G decrease across uniform distribution to Cauchy distribution. Hence, if the process variables are from uniform distribution, G control chart is less sensitive in detecting smaller shifts, whereas, it reflects strong sensitivity to detect shifts under heavier tailed distributions like Laplace and Cauchy distributions. Under exponential distribution, though ARL_G and MRL_G are decreasing for smaller shifts, the detection of shift could be illusive since these values are smaller at zero shift itself.

On comparing the G control chart with median (M) control chart due to Bhat and Patil (2024a), in terms of ARL, we observe that, ARL_G is lower than ARL_M under uniform, normal and logistic distributions, whereas, ARL_G is higher than ARL_M for Laplace distribution. These values are almost same under Cauchy distribution. This indicates that, G control chart detects shifts in the process location quickly than the M control chart when process has uniform, normal, logistic distributions and has almost equal efficiency when process has Cauchy distribution. As G curtails 29% outliers and M curtails 50% outliers, G control chart is less resilient than M control chart to outliers. Here, G control chart is better than M control chart when process variables are from light and medium tailed distributions. However, it is equally competitive when process variables have heavy tailed distributions.

4. Illustration

In this section, we illustrate G control chart with two examples given in Duncan (1955) and Ghute and Shirke (2008).

Example 1: The example due to Duncan (1955) deals with the height (inches) of fragmentation of bomb base that consists of 5 samples taken 29 times and is given in Table 4. The sd, control limits and width of G control chart is given in Table 5. Figure 2 is plotted using Table 4 and Table 5.

Since $\mu(G)$ and σ_G are unknown, they are estimated from the samples. Suppose n samples are taken m times, $\mu(G)$ is estimated by taking the weighted average of 33rd, 50th and 66th percentiles of the samples, with weights respectively given by 0.3, 0.4 and 0.3. That is, $\hat{\mu}(G) = \frac{1}{m}\sum_{i=1}^{m}\left(0.3Q_{\frac{1}{3}}(x_i) + 0.4Q_{\frac{1}{2}}(x_i) + 0.3Q_{\frac{2}{3}}(x_i)\right)$ where $Q_{\frac{1}{3}}(x_i)$, $Q_{\frac{1}{2}}(x_i)$ and $Q_{\frac{2}{3}}(x_i)$ is 33rd, 50th and 66th percentiles of the i^{th} sample. And σ_G is estimated by obtaining $\hat{\sigma}_G$ under various distributions by substituting $\delta\hat{\lambda}$ where $\delta = \sqrt{\frac{n-1}{2}}\left(\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)$ for λ. Here $\hat{\lambda} = \frac{1}{m}\sum_{i=1}^{m} s_i$, where $s_i^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2$. For $n = 5$, $\delta = 1.0638$.

Table 4. Height (inches) of fragmentation of bomb base and computed values of G, s_i

m	n					G	s_i
	1	2	3	4	5		
1	0.831	0.829	0.836	0.840	0.826	0.8316	0.0056
2	0.834	0.826	0.831	0.831	0.831	0.8310	0.0029
3	0.836	0.826	0.831	0.822	0.816	0.8261	0.0078
4	0.833	0.831	0.835	0.831	0.833	0.8326	0.0017
5	0.830	0.831	0.831	0.833	0.820	0.8308	0.0051
6	0.829	0.828	0.828	0.832	0.841	0.8294	0.0055
7	0.835	0.833	0.829	0.830	0.841	0.8328	0.0048
8	0.818	0.838	0.835	0.834	0.830	0.8334	0.0078
9	0.841	0.831	0.831	0.833	0.832	0.8320	0.0042
10	0.832	0.828	0.836	0.832	0.825	0.8312	0.0042
11	0.831	0.838	0.844	0.827	0.826	0.8315	0.0077
12	0.831	0.826	0.828	0.832	0.827	0.8284	0.0026
13	0.838	0.822	0.835	0.830	0.830	0.8310	0.0061
14	0.815	0.832	0.831	0.831	0.838	0.8312	0.0086
15	0.831	0.833	0.831	0.834	0.832	0.8320	0.0013
16	0.830	0.819	0.819	0.844	0.832	0.8281	0.0104
17	0.826	0.839	0.842	0.835	0.830	0.8347	0.0065
18	0.813	0.833	0.819	0.834	0.836	0.8303	0.0103
19	0.832	0.831	0.825	0.831	0.850	0.8312	0.0095
20	0.831	0.838	0.833	0.831	0.833	0.8326	0.0029
21	0.823	0.830	0.832	0.835	0.835	0.8322	0.0049
22	0.835	0.829	0.834	0.826	0.828	0.8298	0.0039
23	0.833	0.836	0.831	0.832	0.832	0.8322	0.0019
24	0.826	0.835	0.842	0.832	0.831	0.8324	0.0059
25	0.833	0.823	0.816	0.831	0.838	0.8298	0.0087
26	0.829	0.830	0.830	0.833	0.831	0.8302	0.0015
27	0.850	0.834	0.827	0.831	0.835	0.8336	0.0087
28	0.835	0.846	0.829	0.833	0.822	0.8326	0.0088
29	0.831	0.832	0.834	0.826	0.833	0.8320	0.0031

Table 5. σ_G, UCL_G, LCL_G and w_G under various distributions

Distribution	σ_G	UCL_G	CL_G	LCL_G	w_G
Uniform	0.0039	0.8430	0.8313	0.8196	0.0235
Exponential	0.0024	0.8370	0.8297	0.8224	0.0146
Normal	0.0030	0.8403	0.8313	0.8223	0.0179
Logistic	0.0027	0.8393	0.8313	0.8233	0.0160
Laplace	0.0020	0.8374	0.8313	0.8252	0.0123
Cauchy	0.0011	0.8346	0.8313	0.8280	0.0066

From Table 5 and Figure 2, we observe that, w_G is narrower for Cauchy distribution, whereas, it is wider for uniform distribution. The centre line of exponential distribution differs from that of symmetric distribution. From Figure 2, we see that, the process is out of control for Cauchy distribution, whereas, it exhibits in control status under other symmetric distributions.

Example 2: The partial data is taken from Ghute and Shirke (2008) which is due to Chen et al. (2005). The data consists of 5 samples taken 18 times from a spring manufacture process. The measurements of the inner diameter of the springs are given in Table 6 and sd, control limits, w_G are given in Table 7. Figure 3 is plotted using Table 6 and Table 7. The computations are carried out on similar lines of example 1.

From Table 7 and Figure 3, we observe that, w_G becomes narrower for heavier tailed distributions. The centre line of all symmetric distributions is the same, whereas, it is different for exponential distribution. From Figure 3, we notice that, process is out of control under Cauchy and Laplace distributions, while it remains in control under other symmetric distributions.

Table 6. Springs inner diameter and computed values of G, s_i

m	n					G	s_i
	1	2	3	4	5		
1	28.14	28.31	28.27	28.20	28.26	28.2497	0.0666
2	28.50	28.35	28.30	28.32	28.20	28.3217	0.1085
3	28.29	28.30	28.29	28.38	28.29	28.2919	0.0394
4	28.22	28.26	28.27	28.27	28.28	28.2680	0.0235
5	28.30	28.36	28.27	28.32	28.30	28.3038	0.0332
6	28.34	28.29	28.32	28.27	28.19	28.2917	0.0581
7	28.24	28.32	28.31	28.36	28.41	28.3256	0.0630
8	28.23	28.36	28.34	28.31	28.33	28.3278	0.0503
9	28.25	28.39	28.31	28.35	28.32	28.3237	0.0518
10	28.31	28.28	28.31	28.36	28.32	28.3119	0.0288
11	28.34	28.31	28.25	28.30	28.45	28.3137	0.0745
12	28.27	28.23	28.35	28.37	28.36	28.3356	0.0623
13	28.35	28.44	28.42	28.32	28.31	28.3573	0.0589
14	28.32	28.30	28.32	28.33	28.40	28.3219	0.0385
15	28.27	28.33	28.41	28.44	28.41	28.3937	0.0701
16	28.35	28.29	28.38	28.35	28.31	28.3418	0.0358
17	28.36	28.38	28.28	28.32	28.40	28.3557	0.0482
18	28.36	28.31	28.38	28.34	28.34	28.3438	0.0261

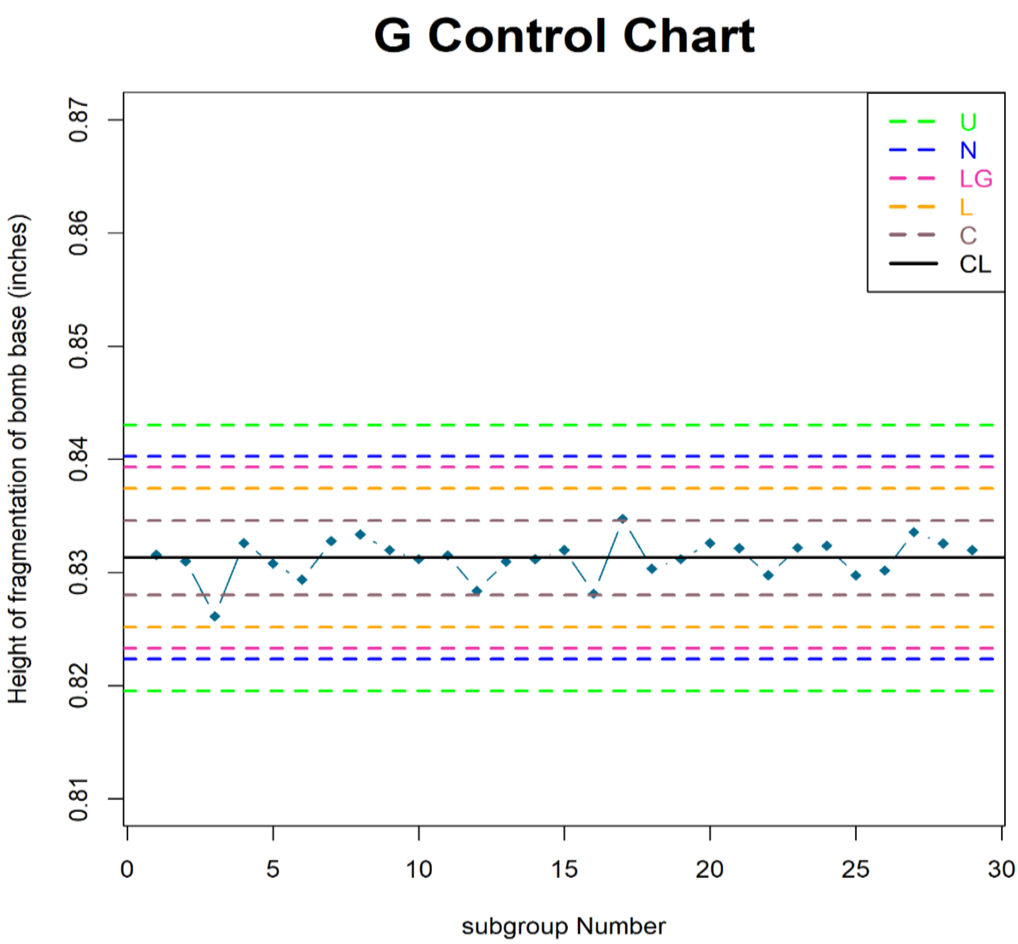

Figure 2. G control chart for symmetric distributions

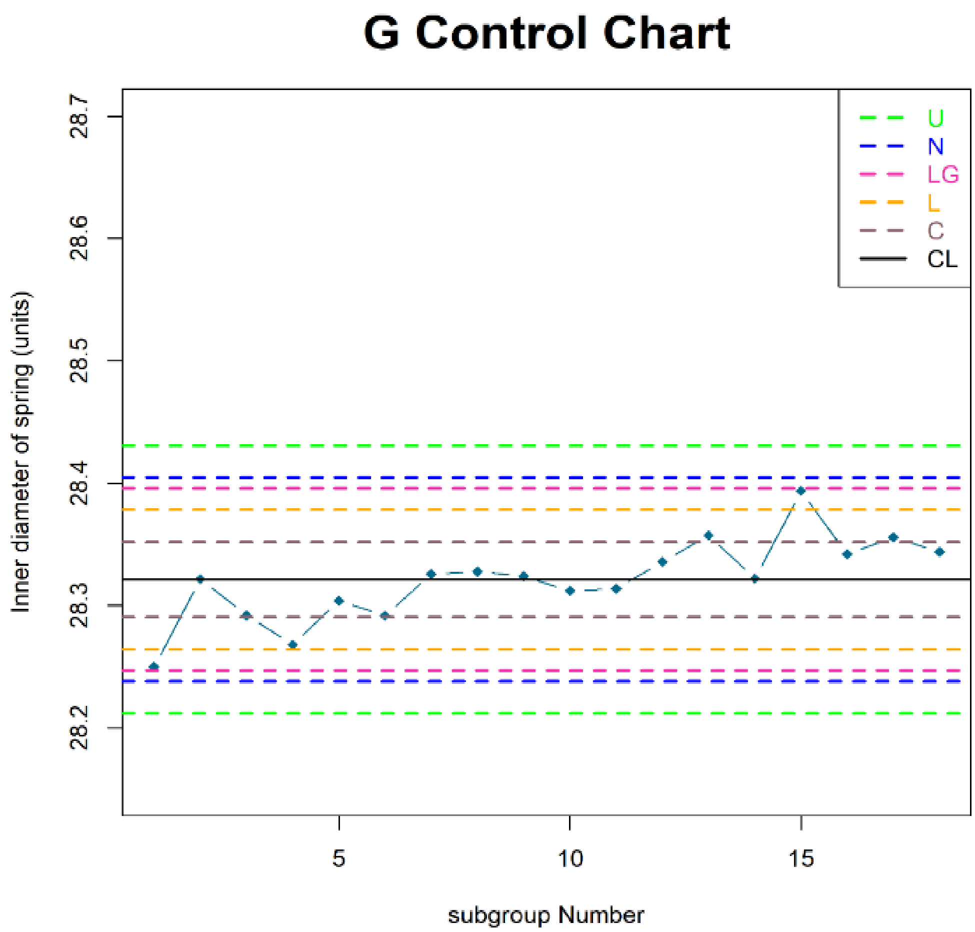

Figure 3. G control chart for symmetric distributions

Table 7. σ_G, UCL_G, LCL_G and w_G under various distributions

Distribution	σ_G	UCL_G	CL_G	LCL_G	w_G
Uniform	0.0364	28.4304	28.3211	28.2118	0.2186
Exponential	0.0226	28.3740	28.3061	28.2381	0.1359
Normal	0.0278	28.4045	28.3211	28.2377	0.1667
Logistic	0.0248	28.3955	28.3211	28.2467	0.1488
Laplace	0.0190	28.3782	28.3211	28.2640	0.1142
Cauchy	0.0102	28.3517	28.3211	28.2905	0.0613

5. Conclusions

In this section, based on our findings, we record our conclusions on the proposed G control chart.

- A control chart based on Gastwirth estimator (G) has been proposed for monitoring process location under the assumption that, the process variables have cumulative density function, $F(x)$.
- The sd of G estimator is obtained and the control limits of the proposed control chart is developed under various symmetric and asymmetric distributions.
- The proposed control chart is evaluated using various performance measures viz. power, ARL, MRL and SDRL for different shifts and sample sizes.
- Power of the G control chart is high for Cauchy distribution as compared to other distributions.
- ARL and MRL decrease for smaller shifts and sample sizes under heavy tailed distributions like Laplace and Cauchy distributions.
- The proposed control chart has lucidity in detecting the shifts in the process location under Cauchy model.
- The G control chart outperforms M control chart due to Bhat and Patil (2024a), when process variables have uniform, normal, logistic distributions and equally performs under Cauchy distribution.
- The proposed control is deplorable under skewed distribution like exponential distribution as the detection of shifts by control chart could be misleading.
- The G control chart is useful when the distributional assumption of normality is not valid. Also, it is highly desirable when process variables are from heavy tailed distributions.

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REFERENCES

[1] Amin, R. W., Reynolds, M. R. and Saad, B. 1995. Nonparametric quality control charts based on the sign statistic. Communications in Statistics - Theory and Methods, 24(6), 1597–1623. https://doi.org/10.1080/03610929508831574.

[2] Arnold, H. J. 1965. Small sample power of the one sample Wilcoxon test for non-normal shift alternatives. The Annals of Mathematical Statistics, 1767-1778. http://dx.doi.org/10.1214/aoms/1177699805.

[3] Bakir, S. T. 2004. A distribution-free Shewhart quality control chart based on signed-ranks. Quality Engineering, 16(4), 613-623. https://doi.org/10.1081/QEN-120038022.

[4] Bhat, S. V. and Patil, S. 2024b. Midrange control chart under non normality. Global Journal of Pure and Applied Mathematics. 20. 169-182. DOI:10.37622/GJPAM/20.1.2024.169-182.

[5] Bhat, S. V. and Patil, S. S. 2024a. Shewhart type control chart based on median for location parameter. International Journal of Agricultural & Statistical Sciences, 20(1).

[6] Chakraborti, S. and Graham, M. 2019. Nonparametric statistical process control. John Wiley & Sons.

[7] Chen, G., Cheng, S. W. and Xie, H. 2005. A new multivariate control chart for monitoring both location and dispersion. Communications in Statistics—Simulation and Computation ®, 34(1), 203-217. DOI: 10.1081/SAC-200047087.

[8] DasGupta, A. 2008. Asymptotic theory of statistics and probability (Vol. 180). New York: Springer.

[9] Duncan, A. J. 1955. Quality control and industrial statistics. RD Irwin.

[10] Gastwirth, J. L. 1966. On robust procedures. Journal of the American Statistical Association, 61(316), 929-948. https://doi.org/10.1080/01621459.1966.10482185.

[11] Ghute, V. B. and Shirke, D. T. 2008. A multivariate synthetic control chart for monitoring process mean vector. Communications in Statistics—Theory and Methods, 37(13), 2136-2148. DOI: 10.1080/03610920701824265.

[12] Harrell, F. E. and Davis, C. E. 1982. A new distribution-free quantile estimator. Biometrika, 69(3), 635-640. https://doi.org/10.1093/biomet/69.3.635.

[13] Hodges, J.L. and Lehmann, E.L. 1963. Estimates of Location Based on Rank Tests. Annals of Mathematical Statistics, 34, 598-611. https://www.jstor.org/stable/2238406.

[14] Montgomery, D. C. 1996. Introduction to statistical quality control (6th Edition ed.). John Wiley & Sons.

[15] Page, E. S. 1961. Cumulative sum charts. Technometrics, 3(1), 1-9. https://doi.org/10.2307/1266472.

[16] Pawar, V. Y. and Shirke, D. T. 2010. A nonparametric Shewhart-type synthetic control chart. Communications in statistics-Simulation and Computation, 39(8), 1493-1505. https://doi.org/10.1080/03610918.2010.503014.

[17] Roberts, S.W. 1959 Control Chart Tests Based on Geometric Moving Averages. Technometrics, 1, 239-250. http://dx.doi.org/10.1080/00401706.1959.10489860.

[18] Shewhart, W. A. 1931. Statistical method from an engineering viewpoint. Journal of the American Statistical Association, 26(175), 262-269.

[19] Zombade, D. M. and Ghute, V. B. 2019. Shewhart-type nonparametric control chart for process location. Communications in Statistics-Theory and Methods, 48(7), 1621-1634. https://doi.org/10.1080/03610926.2018.1435811.