Entailment II

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Abstract We here propose a solution to the problem we have raised in (Pinheiro, 2016). Basically, the mathematical notion of entailment seems to be connected to the inferential rules from Classical Logic, so that if we have *P: x belongs to the reals*, and *Q: x+2=5 => x=3, P |= Q*. Notwithstanding, we would also have that if *P: x belongs to the interval (7,10)*, and *Q: x+2=5 => x=3, P |= Q*. The second instance of entailment does not seem to be justifiable if our intuition is consulted: Even though we could say that absurdity implies anything in Classical Logic, entailment should be a concept that belongs to the metalogic, not to the logic of the system, so that we should not be inside of the Classical Logic World by the time we assess things in what regards entailment. As another point, our discussions in Entailment led us to choose the sense *has as a consequence* for entailment, so that it is not really acceptable that we have, as a consequence, using normal language, of *x* being inside of the real interval *(7,10)* that if *x+2=5*, then *x=3*. If *x* is in that interval, *x+2* should not be *5*, unless we are talking about moduli of vectors. However, when we transfer to Classical Logic, in having the antecedent false, and the consequent false as well, we have that the implication, which some call material, is true. We here discuss this sort of matter in detail, and hope to get to the bottom of the issue, and perhaps to a universal solution.

Keywords Logic, Implication, Entailment, Classical, Nonclassical

1. Introduction

In (Pinheiro, 2016), we discussed the notion of Entailment from several perspectives: From a lexographic point of view, it should probably mean *legal limitation*, what, in what comes to the metalogic of the logical systems, could be translated into *is imposed as a limitation to*. From a logical point of view, considering tradition, it should probably mean *has as consequences* or *has as a consequence*, and if we consider a non-traditional line of thinking, so say Nonclassical Logic, it should probably mean, as for current literature, *whatever makes the antecedent true makes the consequent true*, where antecedent is all that comes before the logical symbol for entailment and consequent is all that comes after it.

We have concluded that the nonclassicists should probably have produced more work in terms of this notion to make it match their new systems but things could be fixed in a subtler manner if they accepted the definition as being *has as a consequence* or *as consequences*, just like the classicists.

In this case, Priest's definition (2001) is not acceptable anymore. We cannot have

> 1.1.5 It is also standard to define two notions of validity. The first is *semantic*. A valid inference is one that *preserves truth*, in a certain sense. Specifically, every interpretation (that is, crudely, a way of assigning truth values) that makes all the premises true makes the conclusion true. We use the metalinguistic symbol '|=' for this. What distinguishes different logics is the different notions of interpretation they employ.

(Pinheiro, 2016b)

We would have to simply drop this definition.

Perhaps we can mention names like Gentzel, Wittgenstein, Russell, and Frege here, but what really matters is that defining Entailment in our Logical Lexicon as *has as a consequence* or *has as consequences* is the best way to address the current problem we have with the definition of the sigmatoid (Pinheiro, 2015) and its current application in Mathematics, therefore Classical Logic, and in other exact sciences, one of those applications being Nonclassical Logic per se.

We here will try to put together the most perfect argument of all to prove our point.

2. Development

Proops (p. 10, 2002) states that

Wittgenstein takes logical entailment to consist in an *internal* relation between the *forms* of propositions, while his opponent takes it to consist in something more— something Wittgenstein characterizes as *[the putting of the propositions] in these relations.... by connecting them with one another in a proposition.*

We cannot make much sense about these words: Internal relation between the forms of propositions. It must have to do however with considering a certain form, so say a conditional, and then seeing if there is a relation between one that we already accept and the one we have at hand. It could also have to do with some internal relation between the antecedent and consequent.

If Wittgenstein (20th century) thought like what we have just described, then his thinking has been abandoned in favour of the thinking of the individual the scholar Proops (2002) called his opponent.

Russell is pointing to the need to distinguish the fact that p materially implies q, from the fact that q is logically entailed by, or in his terminology "deducible from," p. He defines the latter notion as follows.

We may then say that *q* is *deducible* from *p* if it can be shown *by means of the above principles* [i.e. the axioms and inference rules of Russell's system] that *p* implies *q*. (Ibid.)

So q is deducible from p just in case 'p⊃q' is derivable in Russell's system.

Russell observes that the definition captures the extension of the concept he is seeking to define:

This meaning of *deducible* is purely logical, and covers, I think, exactly the cases in which, in practice, we can deduce a proposition *p* from a proposition *p* without assuming either that *p* is false or that *q* is true. (Ibid.)

(Proops, p. 13, 2002)

We here see an attempt to form a strong relationship between material implication and entailment. As we noticed before, in (Pinheiro, 2016), this will imply that antecedent false and consequent true gives us a deduction, so that anything you want to write in the part we are calling antecedent will constitute a deduction for the right side, even false claims.

That idea was obviously dropped as we evolved in reasoning in terms of entailment, like it is OK to have a false antecedent leading to a true consequent getting an OK from us in what regards the material implication, but not in what regards the meta language, not when we need to justify why we are accepting something as a derivation or consequence of our system: If it is supposed to be a consequence of the paradigms or axioms we have in our system, then it has to actually derive from the assumption that those are true, not false.

A Fregean view that might be the target of Wittgenstein's criticism is suggested in a passage from "Foundations of Geometry: Second Series," where Frege is discussing, not logical entailment exactly, but the related notion of the "dependence" of one thought on a group of others.[34] He writes:

> Let Ω be a group of true thoughts. Let a thought G follow from one or several of the thoughts of this group by means of a logical inference such that apart from the laws of logic, no proposition not belonging to Ω is used. Let us now form a new group of thoughts by adding the thought G to the group Ω. Call what we have just performed a logical step. Now if through a sequence of such steps, where every step takes the result of the preceding one as its basis, we can reach a group of thoughts that contains the thought A, then we call A dependent upon group Ω. If this is not possible, then we call A independent of Ω.[35]

(Proops, p. 16, 2002)

We agree with Proops (2002): Frege seems to be talking about entailment. In this case, we notice that he indicates connection in terms of meaning between our antecedent and our consequent in the entailment relationship. Such a meaning is novelty because we just demand the truth of the antecedent to come together with the truth of the consequent when the interpretation we apply to the antecedent is used also for the consequent: We do not demand connection in terms of meaning. In this way, even zero premises to the left with something that is true in every interpretation to the right gives us entailment: No basis for thoughts should be found in zero premises however.

It does not seem reasonable to require that we now have something to the left for us to accept a tautology in the system, so that we shouldn't really try to do that with entailment, but it could be worth it exploring the possibility when it comes to the material condition, and some people have already tried to do that, as we know.

A constraint in a formal language classification is a sequent, in the sense of Gentzen's sequent calculus, and the relation $\vdash_L$ is Gentzen's entailment relation, defined semantically in the Tarskian manner: $\Gamma \vdash_L \Delta$ iff every token that satisfies every formula in Γ also satisfies at least one formula in Δ.[24] With this notation, an information flow in a Tarski channel with interpretations f and g from languages L_1 and L_2, each into interlingua L, can also be characterised as follows:

Information flow in a Tarski channel Formula φ_2 of language L_2 carries the information expressed by formula φ_1 of L_1 (given K) iff $K, g^{\wedge}(\varphi_2) \vdash_L f^{\wedge}(\varphi_1)$ but neither $K \vdash_L f^{\wedge}(\varphi_1)$ nor $Kg^{\wedge}(\varphi_2) \vdash_L$.

(Sommaruga, 2009)

It looks like Gentzen's idea of entailment had to do with supporting, since tao would generate at least one formula in delta through that interpretation that satisfies all in tao, but would not have to satisfy all delta, not even two of its formulae. It is like whatever is to the left side supports in part whatever is to the right, but supports at least one formula in whatever is to the right.

This seems to have nothing in common with the etymological meaning of entailment, which we exposed in (Pinheiro, 2016). It also has nothing in common with whatever we mention there, in (Pinheiro, 2016), in terms of definition of entailment.

Tarski, however, according to (Sommaruga, 2009), could have in common in terms of thinking and entailment, in common with Gentzen. See:

Following Information Via Logic instead of Information Via Probability, a similar model of information can be stated using Tarski channels. In a Tarski channel, information flows from one language to the other via interpretation in a common language (the interlingua). Given a message in the language of L_2 we can learn about events as described in L_1 by translating between the two languages. To determine what information is carried by a formula φ_2 of L_2, it can be translated into L_1. The information expressed by formulas φ_1 of L_1 is carried if it is entailed by a translation of φ_2. Translation in a Tarski channel occurs using the interlingua, by interpreting both φ_2 and φ_1 into the interlingua and then seeing whether there is an entailment between them. By analogy with Dretske's definition, consideration can be given to the agent's knowledge, which can be represented as a set K of interlingua formulas, and the borderline cases in which φ_2 is already known or φ_1 contradicts current knowledge can both be deemed not to count as cases of information flow. Summarising: in a Tarski channel, formula φ_2 of language L_2 can be said to carry the information expressed by formula φ_1 (given K) if and only if the interlingua translation of φ_1 is entailed by K together with the translation of φ_2, excluding the case that φ_1 is already a consequence of K and the case that φ_2 contradicts K.

The role played by probability in a Shannon channel is now played by entailment, which is defined semantically: φ entails ψ iff every structure m of type φ is also of type ψ. The definition of entailment depends only on the satisfaction relation $\models$, and so can be formulated in the language of classifications alone.

After reading (Sommaruga, 2009), the extract above, we reach the conclusion that Tarski seemed to see entailment in a very different way, so that he also seems not to agree with Gentzen.

Tarski seems to believe in a language that is in-between two other languages, an interface, let's say. In this case, we would translate a formula from the first language into this interface language. We would also translate the formula that is supposed to be entailed from the second language into the interface language. We have the knowledge of the agent, which is being called K, to consider as well. We then need to know if K entails this formula from the first language, after it is translated into the interface language, as well as the formula we analyse in the second language, also after it is translated into the interface language, and we block the cases in which the first formula is already a logical consequence of K and the cases in which the formula from the second language contradicts K when doing this. If such a thing happens, then we say that the formula from the second language does carry the information from the formula we chose from the first language.

Tarski's definition of entailment then looks similar to the definition we currently have for Classical Logic: If we replace K with interpretation, it should end up being the same or almost the same. If our interpretation entails what is to the left as well as what is to the right, then we have an entailment from left to right in terms of that interpretation. If all interpretations that entail what is to the left also entail what is to the right, then we have an entailment in general from left to right. Perhaps, if we could use the language we see in the other paragraph, we would then say that if the knowledge of every agent entails both the left and the right side, we have an entailment in general from left to right.

It a sort of makes sense having the interpretation of every agent, which has to be based on their knowledge, being used to decide on whether we have an entailment inside of that system or not. From Tarski, we also get the idea of going from left to right: Whatever interpretation makes the antecedent true has to make the consequent true as well for us to have an entailment.

We believe that entailment should always be considered in context. In the same way that we say that the knowledge of the agent entails whatever is to the left, what would then be their interpretation if we take for granted that we all accept our proposal, we have to consider that entailment only happens inside of the system we analyse, like we have no other way to go about it.

If we always thought of entailment as being something connected to Classical Logic, as Priest seems to do, we would be using Classical Logic to decide on possible entailments inside of the new systems, so that we would be using it as a metalogic.

Notwithstanding, a metalogic should be more encompassing than the logical system, which is closer to the machine, not more limited. If we have Fuzzy Logic as our logical system and we use Classical Logic to make the higher order decisions for Fuzzy Logic, it just seems irrational not to say that the logical system is the metalogic instead, since that seems to imply saying that Classical Logic is more encompassing than Fuzzy Logic, but this seems to be a contradiction, so that we cannot really do that.

In that being our decision, our metalogic should probably always be more encompassing than our logical system. It should not have fewer inclusions, that is for sure.

Oh, well, in not being able to draw something more encompassing than all logical systems ever proposed, we should choose the same level of inclusion for the technical part of it, and therefore the same logical system. Even so, the metalogic shall be more encompassing because it will accept, for instance, all sigmatoids from the purely human language.

With this, we will actually stick to the choice we had in (Pinheiro, 2016), and entailment shall be defined in the following way:

> A set of formulas F logically entails G (or G is a logical consequence of F or G is a theorem of F), in symbols F |= G, iff each model for F is also a model for G.
>
> (Brandeis, 2008)

Here we can once more see that part of the literature already tells us that entails should be equated to *makes us have, as a logical consequence,* or *has, as a logical consequence,* or *has as logical consequence.*

F |= G iff each model for F is also a model for G is actually equivalent to the definition adopted by Priest for Nonclassical Logic (2001), since that implies that all interpretations that make F true make G true in Classical Logic.

What follows *implies* is the part that has to be changed in Nonclassical Logic however.

As said in (Pinheiro, 2016), if the logical system has anything else as truth-values, so say cool and not-cool, this definition is not OK anymore.

What we want is basically keeping what is metalogic inside of the World of Metalogic and what is logical system outside of this world.

We could still use the words we have just seen, so that *F |= G iff each model for F is also a model for G*, but we would then have to change the definition of model.

Model will now be whatever makes F be included in the set of accepted propositions of the system we consider.

In the metalogic, we should avoid the symbols and the specific language (say true and false) we use inside of the particular logical system we analyse/describe for obvious reasons. One of those is to avoid confusion, such as the one we have just seen (the definition of entailment used in Classical Logic has been adopted for all nonclassical systems).

When we describe a logical system, we tell how we are making propositions be accepted/included, so say via assigning them values such as true and possible or numbers such as 1 and 0.5.

So, say we have Fuzzy Logic, the system created by Zadeh:

> Fuzzy logic is an approach to computing based on *degrees of truth* rather than the usual *true or false* (1 or 0) Boolean logic on which the modern computer is based.
>
> The idea of fuzzy logic was first advanced by Dr. Lotfi Zadeh of the University of California at Berkeley in the 1960s. Dr. Zadeh was working on the problem of computer understanding of natural language. Natural language (like most other activities in life and indeed the universe) is not easily translated into the absolute terms of 0 and 1. (Whether everything is ultimately describable in binary terms is a philosophical question worth pursuing, but in practice much data we might want to feed a computer is in some state in between and so, frequently, are the results of computing.) It may help to see fuzzy logic as the way reasoning really works and binary or Boolean logic is simply a special case of it.
>
> Fuzzy logic includes 0 and 1 as extreme cases of truth (or *the state of matters* or *fact*) but also includes the various states of truth in between so that, for example, the result of a comparison between two things could be not *tall* or *short* but *.38 of tallness*.
>
> Fuzzy logic seems closer to the way our brains work. We aggregate data and form a number of partial truths which we aggregate further into higher truths which in turn, when certain thresholds are exceeded, cause certain further results such as motor reaction. A similar kind of process is used in neural networks, expert systems and other artificial intelligence applications. Fuzzy logic is essential to the development of human-like capabilities for AI, sometimes referred to as artificial general intelligence: The representation of generalized human cognitive abilities in software so that, faced with an unfamiliar task, the AI system could find a solution.
>
> (Rouse, 2016)

In our opinion, logical systems should be designed for a certain purpose, so that the motivation for the creation of each new system would have to be the purpose, the problem at hand. In this way, we could imagine that Hyde were building a system to cope with his demands in what regards the issues raised by The Sorites Paradox (Pinheiro, 2016c), and his concerns were then limited to the application of the lexicon sigmatoids to the human discourse in terms of the function adjective. One could argue that this is grammar not syntax, and syntax is the sigmatoid we chose to designate function in language, but there is still a very specific role played by the sigmatoids we here refer to, and we are then calling this role function.

So, let's imagine that we can solve Hyde's problem by adopting a version of Fuzzy Logic. In this case, bald is a person who gets assigned more than 0.5 degrees of baldness. The same applies to all sigmatoids that are or could be classified as adjectives in his system. Hyde's system is now Fuzzy Logic for adjectives or propositions containing those plus Classical Logic for everything else.

Our metalogic would still say that Entailment is

A set of formulas F logically entails G (or G is a logical consequence of F or G is a theorem of F), in symbols F |= G, iff each model for F is also a model for G,

as said before, but our logical system, that is, in this case, Hyde's system, which is a version of Fuzzy Logic in what regards adjectives or propositions containing those plus Classical Logic for everything else, would say that a model is defined as in Classical Logic for F in case the proposition does not contain adjectives, and as in Fuzzy Logic with the extra rule (more than 0.5 is true, less or equal is false) in case the proposition contains those.

This logic would be equal to Classical Logic when there is no adjective or term that works as an adjective in any of the propositions involved, and would be equal to Hyde's version of Fuzzy Logic otherwise.

To prove completeness and soundness, Priest's style, we would have to use the proof for Classical Logic when there are no adjectives or terms that work as if they were adjectives, and the proof for Fuzzy Logic, Hyde's version, otherwise. In this case, we would have both soundness and completeness getting a tick of approval.

3. Conclusions

We concluded that we can preserve the definition of entailment we most liked in (Pinheiro, 2016), which was

A set of formulas F logically entails G (or G is a logical consequence of F or G is a theorem of F), in symbols F |= G, iff each model for F is also a model for G,

but we have to remember to then define *model* for each, and every, new logical system we create.

Model will now be whatever makes F be included in the set of accepted propositions of the system we consider.

Upon thinking once more about what Hyde seemed to be wishing for in terms of The Sorites Paradox, we theorized that he could feel content with a logical system that were a mix between Classical Logic and a version of Fuzzy Logic, where Fuzzy Logic would be applied exclusively in situations in which we had propositions involving adjectives or terms that are being used as if they are adjectives.

In this case, instead of *I am bald* receiving a false or true as a tag, it would receive a real number between 0 and 1, so say 0.7. Since 0.7 is bigger than 0.5, where the line was drawn, that would mean that the proposition *I am bald* is true.

We do have to abandon the current definition of entailment, which is that whatever interpretation that makes the antecedent true makes the consequent true equates entailment.

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