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Conservation of Energy and Faraday Tensor in Transformation of Electromagnetic Fields, Maxwell's Equations, and Symmetric Electromagnetic Field Energy-Momentum Tensor in Theory of Lorentz Invariant Relativistic Electrodynamics

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Abstract We propose a new way of using law of energy conservation and Faraday tensor for transformation of electromagnetic fields between inertial frames in a four-dimensional Minkowski spacetime wherein space and time are linearly inter-related. We prove that the electromagnetic fields are Lorentz invariant for circular boost along the direction of boost and in perpendicular directions and for a spatial rotation of planes relative to a fixed coordinate axis between two inertial frames. We further demonstrate that the product of covariant and contravariant Faraday tensors and of their duals lead to non-zero electromagnetic field Lagrange density in free space. We also derived Lorentz invariant analytical expressions for Maxwell's equations, current continuity equation and symmetric electromagnetic field energy-momentum tensor with and without charge and current source. We further demonstrate that using symmetric electromagnetic field energy-momentum tensor one can reliably derive expressions for Lorentz invariant of electromagnetic fields between two inertial frames in case of circular Lorentz boost and spatial (planar) rotation about a fixed coordinate axis. We believe that the proposed theory may have profound effect on the study of contemporary issues in theory of the relativistic electrodynamics and general relativity.

Keywords Faraday tensor, Conservation of energy, Lorentz invariance, Electromagnetic fields, Maxwell equations, Electromagnetic field energy-momentum tensor, Gravitational field equation

1. Introduction

Maxwell's equations are the foundation of electrodynamics, and they relate electric and magnetic fields to each other [1]. Historically, Lorentz used Maxwell's equations to derive his space-time coordinate transformation rule [2]. Einstein in 1905 used Lorentz coordinate transformation rule to prove the validity of Maxwell's equations in all inertial frames by using two postulates [3]; (i) The physics laws are invariant between two inertial frames. (ii) The speed of light is constant and independent of the direction of the motion of the emitting body in all inertial frames in free space. In electrodynamics, Maxwell's equations and Lorentz force describe how the charge and current sources with densities ρ and $\vec{J}$ generate electric and magnetic fields ($\vec{E}$ and $\vec{B}$) [1]:

$\vec{\nabla}\cdot\vec{E}=\rho/\varepsilon_\circ$: Gauss law of electrostatics;

$$\vec{\nabla}\cdot\vec{B}=0 : \text{Gauss law of magnetism} \quad (1)$$

$\vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$: Faraday's law;

$$\vec{\nabla}\times\vec{B}=\mu_\circ\vec{J}+\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} : \text{Ampere-Maxwell equation} \quad (2)$$

$$\vec{F}=q\vec{E}+q\vec{v}\times\vec{B} : \text{Lorentz force} \quad (3)$$

Maxwell's equations lead to several conservation laws [4], such as current continuity equation, conservation of electromagnetic energy, and momentum for which the electric and magnetic fields are Lorentz invariant between two inertial frames. Proof of Lorentz invariance of electromagnetic fields are interest over almost a century [5]-[8]. Einstein used the rates of momentum $(d\vec{p}/dt)$ and of energy (dE/dt) to transform electric and magnetic fields between two inertial frames [4]

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$$
\begin{aligned}
&E'_x = E_x, \quad E'_y = \gamma\left(E_y + \beta B_z\right), \quad E'_z = \gamma\left(E_z - \beta B_y\right),\\
&B'_x = B_x, \quad B'_y = \gamma\left(B_y - (\beta / c)E_z\right), \quad B'_z = \gamma\left(B_z + (\beta / c)E_y\right),
\end{aligned} \tag{4a}
$$

$$
\begin{aligned}
&E_x = E'_x, \quad E_y = \gamma\left(E'_y - \beta B'_z\right), \quad E_z = \gamma\left(E'_z + \beta B'_y\right),\\
&B_x = B'_x, \quad B_y = \gamma\left(B'_y + (\beta / c)E'_z\right), \quad B_z = \gamma\left(B'_z - (\beta / c)E'_y\right),
\end{aligned} \tag{4b}
$$

where $\gamma = 1(1-\beta^2)^{-1/2}$ is Lorentz factor with velocity normalized to speed of light $(\beta = v/c)$.

The electromagnetic field transformation between two inertial frames is also studied by using the electromagnetic field tensor, which is also called Faraday tensor [4]. In this formalism, one uses covariant and contravariant Faraday tensors $F_{\alpha\beta} = F(\vec{E}, \vec{B})$ and $F^{\alpha\beta} = F(-\vec{E}, \vec{B})$, and their duals $G_{\alpha\beta} = G(\vec{E}, \vec{B})$ and $G^{\alpha\beta} = G(-\vec{E}, \vec{B})$. To obtain the components of $F^{\alpha\beta}$ one uses $\vec{E} \to -\vec{E}$ in $F_{\alpha\beta}$. Likewise, components of dual tensors $G_{\alpha\beta}$ and $G^{\alpha\beta}$ are obtained by using $\vec{E}/c \rightleftarrows \vec{B}$ and $\vec{B} \rightleftarrows -\vec{E}/c$ in $F_{\alpha\beta}$ and $F^{\alpha\beta}$, respectively. For boost along the x-axis, Faraday tensor $F'_{\alpha\beta}$ in frame $\bar{\bar{\Sigma}}'$ is obtained from the classical transformation $F'_{\alpha\beta,z} = \Lambda_z F_{\alpha\beta} \tilde{\Lambda}_z$, explicitly written as

$$
\begin{pmatrix} 0 & E'_x/c & E'_y/c & E'_z/c \\ -E'_x/c & 0 & -B'_z & B'_y \\ -E'_y/c & B'_z & 0 & -B'_x \\ -E'_z/c & -B'_y & B'_x & 0 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5}
$$

Matching both sides of Eq. (5) one finds Eq. (4a) for Cartesian components of electric and magnetic fields in the inertial frame $\bar{\bar{\Sigma}}'$. Likewise, inverse transformation $F_{\alpha\beta,z} = \tilde{\Lambda}_z F'_{\alpha\beta} \Lambda_z$ gives the Cartesian components of covariant tensor $F_{\alpha\beta}$ in frame $\bar{\bar{\Sigma}}$, which then yields Eq. (4b).

The transformed field equations (4a) and (4b) state that: (i) the electric (magnetic) field are Lorentz invariant along the x-boost direction but not in the y and z directions. (ii) The scalar product of electric and magnetic fields is Lorentz invariant $(\vec{E}' \cdot \vec{B}' = \vec{E} \cdot \vec{B})$ and (iii) the vector product of electric and magnetic fields is not Lorentz invariant $(\vec{E}' \times \vec{B}' \neq \vec{E} \times \vec{B})$, contradicting Lorentz transformation of vector quantity which must be invariant between two frames $\bar{\bar{\Sigma}}'$ and $\bar{\bar{\Sigma}}$.

In the classical Faraday tensor transformation, we observe that (i) Trace of the products of Faraday tensors and their duals is $Tr(F_{\alpha\beta} F^{\alpha\beta}) = B^2 - E^2/c^2 = Tr(G_{\alpha\beta} G^{\alpha\beta})$, which has no physical meaning since it yields zero electromagnetic field Lagrange density ($L_{em} = -F_{\alpha\beta} F^{\alpha\beta} / 4\mu_\circ = 0$) in free space, which is not realistic because electromagnetic waves transfer energy and momentum [4]. (ii) Furthermore, the trace of covariant and contravariant Faraday tensor and its dual is equal to $Tr(F_{\alpha\beta} G^{\alpha\beta}) = Tr(F^{\alpha\beta} G_{\alpha\beta}) = 4\vec{E} \cdot \vec{B}/c$, which suggests that Faraday tensor and its dual are orthogonal when scalar product of electric and magnetic fields is zero.

Following the work of Mignani and Recami [9], we recently proposed a 6-dimensional spacetime (3+3) frame [10,11] in which the transformed relativistic velocity is combined with energy conservation to successfully demonstrate the Lorentz invariance of electric and magnetic fields and Maxwell's equations between two frames under rotation. In this work, we extend our recent study [10], [11] to study Lorentz invariance of relativistic quantities (e.g.position, velocity, momentum, force, electromagnetic fields, Poynting vector, Maxwell's equations, and energy-momentum tensor in a four-dimensional spacetime in which both space and time coordinates are linearly inter-related. The outline of our presentation is as follows: In sections 2 and 3 we derive the metric equation and transform 4-velocity, 4-momentum and 4-force vector components between two frames. In sections 4 and 5 we use 4-vector velocity with the law of conservation of energy to prove the invariance of electric and magnetic fields under spatial rotation. In sections 6, 7, 8, and 9, we then combine the law of conservation of energy with Faraday tensor and its dual to study Lorentz invariance of electromagnetic fields, Maxwell equations, current continuity equation, and electromagnetic field energy-momentum tensor between two reference frames under a boost and spatial rotation. In section 10, we discuss the details and applications of the proposed theory in invariant relativistic electrodynamics and gravitational field theory.

2. Generalized Four-Dimensional Spacetime

We introduce a generalized four-dimensional spacetime wherein two "massive inertial frames" $\bar{\Sigma}' = \bar{\Sigma}'(x', y', z', ic't')$ and $\bar{\Sigma} = \bar{\Sigma}(x, y, z, ict)$, which coincide with a stationary inertial frame $\Sigma_0 = \Sigma_0(x_0, y_0, z_0, t_0)$ at time $t' = t = t_\circ = 0$, and move relative to each other with an arbitrary velocity $\vec{v} = (v_x, v_y, v_z)$. The space and time coordinates are inter-related: $x' = x'(x,t)$, $y' = y'(y,t)$, $z' = z'(z,t)$, $t' = t'(t,x,y,z)$ in frame $\bar{\Sigma}'$ and $x = x(x',t')$, $y = y(y',t')$, $z = z(z',t')$, and $t = t(t',x',y',z')$ in frame $\bar{\Sigma}$. We require that Einstein's two postulates are also valid in generalized 4-dimensional spacetime. We consider an event sending a light signal from the origin and second event of arrival at some points $P(x,y,z,ict)$ and $P'(x',y',z',ic't')$ in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$. The square of the displacements are described by

$$ds'^2 = dx'^2 + dy'^2 + dz'^2 - c'^2 dt'^2, \tag{6a}$$

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2, \tag{6b}$$

where the differential space-time displacements are defined as

$$dx' = dx + ic\beta_x dt, \quad dy' = dy + ic\beta_y dt \quad dz' = dz + ic\beta_z dt, \quad dt' = dt + \frac{i}{c}\left(\beta_x dx + \beta_y dy + \beta_z dz\right), \tag{7a}$$

$$dx = dx' + ic'\beta_x dt', \quad dy = dy' + ic'\beta_y dt', \quad dz = dz' + ic'\beta_y dt', \quad dt = dt' + \frac{i}{c'}\left(\beta_x dx' + \beta_y dy' + \beta_z dz'\right) \tag{7b}$$

where $\beta_x = \beta \cos\phi \sin\theta$, $\beta_y = \beta \sin\phi \sin\theta$, $\beta_z = \beta \cos\theta$ in spherical coordinates with $\beta = v/c$.

A pair of events with zero (null) separation connected by a signal at constant speed is described by

$$ds^2 = \gamma_{\mu\nu}^2 ds'^2 = \gamma_{\mu\nu}^2 dx'^\mu dx'^\nu = \gamma^{2\mu\nu} dx'_\mu dx'_\nu, \tag{8}$$

where $\gamma_{\mu\nu} = \gamma^{\mu\nu}$ is generalized Minkowski metric tensor. Equation (8) can be written as

$$\begin{aligned} dx^2 + dy^2 + dz^2 - c^2 dt^2 &= \gamma_{\mu\nu}^2 (dx'^2 + dy'^2 + dz'^2) - \gamma_{\mu\nu}^2 c'^2 dt'^2 \\ &= \gamma_{xx}^2 (1-\beta_x^2) dx^2 + \gamma_{yy}^2 (1-\beta_y^2) dy^2 + \gamma_{zz}^2 (1-\beta_z^2) dz^2 - c'^2 \gamma_{tt}^2 (1-\beta^2) dt^2 - \frac{2i}{c'^2} \delta_{x_i v_i}, \end{aligned} \tag{9}$$

where $\delta_{x_i v_i} = \left(\gamma_{xy}^2 v_x v_y xy + \gamma_{yz}^2 v_y v_z yz + \gamma_{xz}^2 v_x v_z xz\right)$ and $c' = c$. Matching both sides of Eq. (9) gives

$$\gamma_{\mu\nu} = \gamma^{\mu\nu} = \begin{pmatrix} \gamma_{xx} & 0 & 0 & 0 \\ 0 & \gamma_{yy} & 0 & 0 \\ 0 & 0 & \gamma_{zz} & 0 \\ 0 & 0 & 0 & \gamma_{tt} \end{pmatrix} = \begin{pmatrix} (1-\beta_x^2)^{-1/2} & 0 & 0 & 0 \\ 0 & (1-\beta_y^2)^{-1/2} & 0 & 0 \\ 0 & 0 & (1-\beta_z^2)^{-1/2} & 0 \\ 0 & 0 & 0 & (1-\beta^2)^{-1/2} \end{pmatrix}, \tag{10}$$

as generalized Lorentz scaling factor, with $\beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2 = v^2/c^2$ and. $\gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$

Figure 1 shows that γ_{xx}, γ_{yy} are anisotropic and γ_{tt} is uniform at any azimuthal angle ϕ.

The differential space and time coordinate displacements are then written in matrix form

$$\begin{pmatrix} dt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} \gamma_{tt} & 0 & 0 & 0 \\ 0 & \gamma_{xx} & 0 & 0 \\ 0 & 0 & \gamma_{yy} & 0 \\ 0 & 0 & 0 & \gamma_{zz} \end{pmatrix} \begin{pmatrix} d\bar{t} \\ d\bar{x} \\ d\bar{y} \\ d\bar{z} \end{pmatrix}, \tag{11a}$$

$$\begin{pmatrix} dt \\ dx \\ dy \\ z \end{pmatrix} = \begin{pmatrix} \gamma_{tt} & 0 & 0 & 0 \\ 0 & \gamma_{xx} & 0 & 0 \\ 0 & 0 & \gamma_{yy} & 0 \\ 0 & 0 & 0 & \gamma_{zz} \end{pmatrix} \begin{pmatrix} d\bar{t}' \\ d\bar{x}' \\ d\bar{y}' \\ d\bar{z}' \end{pmatrix} \tag{11b}$$

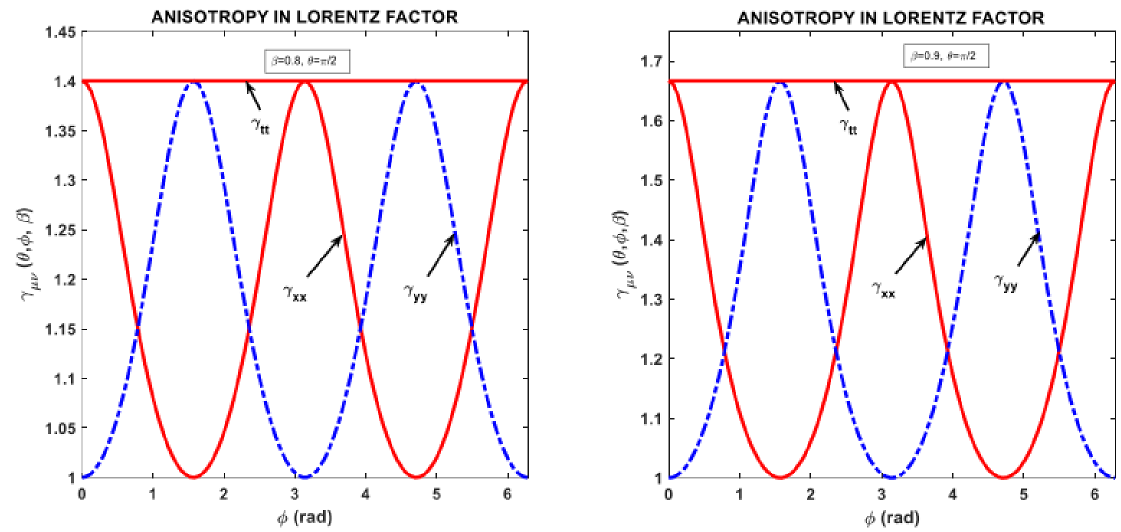

Figure 1. Cartesian components of Lorentz factor as a function of azimuthal angle ϕ in spherical coordinates for polar angle $\theta = \pi/2$ and normalized speed $\beta = 0.8$ (left) and $\beta = 0.9$ (right)

where $d\overline{t} = dt - i(\beta_x dx + \beta_y dy + \beta_z dz)/c$, $d\overline{x} = dx + ic\beta_x dt$, $d\overline{y} = dy + ic\beta_y dt$, $d\overline{z} = dz + ic\beta_z dt$, in frame $\overline{\Sigma}$ and $d\overline{t}' = dt' - i(\beta_x dx' + \beta_y dy' + \beta_z dz')/c'$, $d\overline{x}' = dx' + ic'\beta_x dt'$, $d\overline{y}' = dy' + ic'\beta_y dt'$, and $d\overline{z}' = dz' + ic'\beta_z dt'$ in frame $\overline{\Sigma}'$. Equations (11a) and (11b) is rewritten in a familiar matrix form as

$$\begin{pmatrix} dt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} \gamma_{tt} & i\gamma_{tt}\beta_x & i\gamma_{tt}\beta_y & i\gamma_{tt}\beta_z \\ -i\gamma_{xx}\beta_x & \gamma_{xx} & 0 & 0 \\ 0 & -i\gamma_{yy}\beta_y & \gamma_{yy} & 0 \\ 0 & 0 & -i\gamma_{zz}\beta_z & \gamma_{zz} \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ z \end{pmatrix}; \quad \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \gamma_{tt} & -i\gamma_{tt}\beta_x & -i\gamma_{tt}\beta_y & -i\gamma_{tt}\beta_z \\ i\gamma_{xx}\beta_x & \gamma_{xx} & 0 & 0 \\ 0 & i\gamma_{yy}\beta_y & \gamma_{yy} & 0 \\ 0 & 0 & i\gamma_{zz}\beta_z & \gamma_{zz} \end{pmatrix} \begin{pmatrix} dt' \\ dx' \\ dy' \\ dz' \end{pmatrix}, \tag{12}$$

The space and time differential displacement equations (7a) and (7b) are then replaced with

$$\begin{aligned} &dx' = \gamma_{xx}(dx - ic\beta_x dt), \quad dy' = \gamma_{yy}(dy - ic\beta_y dt), \quad dz' = \gamma_{zz}(dz - ic\beta_z dt), \\ &dt' = \gamma_{tt} dt + i\gamma_{tt}\left(\beta_x dx + \beta_y dy + \beta_z dz\right)/c, \end{aligned} \tag{13a}$$

$$\begin{aligned} &dx = \gamma_{xx}(dx' - ic\beta_x dt'), \quad dy = \gamma_{yy}(dy' - ic\beta_y dt'), \quad dz = \gamma_{zz}(dz' - ic\beta_y dt'), \\ &dt = \gamma_{tt} dt' - i\gamma_{tt}\left(\beta_x dx + \beta_y dy + \beta_z dz\right)/c, \end{aligned} \tag{13b}$$

Using Eqs. (13a) and (13b) one then writes Cartesian components of velocity in frames $\overline{\Sigma}'$ and $\overline{\Sigma}$

$$u'_x = \frac{dx'}{dt'} = \frac{(\gamma_{xx}/\gamma_{tt})(u_x - v_x)}{\left(1 - \left(u_x v_x + u_y v_y + u_z z_z\right)/c^2\right)}, \qquad u_x = \frac{dx}{dt} = \frac{(\gamma_{xx}/\gamma_{tt})(u'_x + v_x)}{\left(1 + \left(u_x v_x + u_y v_y + u_z z_z\right)/c^2\right)}, \tag{14a}$$

$$u'_y = \frac{dy'}{dt'} = \frac{(\gamma_{yy}/\gamma_{tt})(u_y - v_y)}{\left(1 - \left(u_x v_x + u_y v_y + u_z z_z\right)/c^2\right)}, \qquad u_y = \frac{dy}{dt} = \frac{(\gamma_{yy}/\gamma_{tt})(u'_y + v_y)}{\left(1 + \left(u_x v_x + u_y v_y + u_z z_z\right)/c^2\right)}, \tag{14b}$$

$$u'_z = \frac{dz'}{dt'} = \frac{(\gamma_{zz}/\gamma_{tt})(u_z - v_z)}{\left(1 - \left(u_x v_x + u_y v_y + u_z z_z\right)/c^2\right)}, \qquad u_z = \frac{dz}{dt} = \frac{(\gamma_{zz}/\gamma_{tt})(u'_z + v_z)}{\left(1 + \left(u_x v_x + u_y v_y + u_z z_z\right)/c^2\right)}, \tag{14c}$$

Combining Eqs. (13a) and (13b) reduces the four unknowns in Eqs. (14a), (14b) and (14c) to one

$$u'_{x_i} = \frac{dx'_i}{dt'} = \gamma_{x_i x_i}\frac{dx_i}{dt'} - \gamma_{x_i x_i} v_{x_i}\frac{dt}{dt'} = \gamma_{x_i x_i}\frac{dx}{dt'} - \frac{\gamma_{x_i x_i}}{\gamma_{tt}} v_{x_i} + \frac{\gamma_{x_i x_i} v_x}{c^2}\sum_{x_i} v_{x_i}\frac{dx_i}{dt'}, \tag{15a}$$

$$\frac{dx}{dt'} = \gamma_{xx} u'_x + \gamma_{xx} v_x, \qquad \frac{dy}{dt'} = \gamma_{yy} u'_y + \gamma_{yy} v_y, \qquad \frac{dz}{dt'} = \gamma_{zz} u'_z + \gamma_{zz} v_z, \tag{15b}$$

The Cartesian components of the velocity vector $\vec{u}'$ in frame $\bar{\Sigma}'$ are then written as

$$u'_x = -\frac{(\gamma_{xx}/\gamma_{tt}) - \gamma_{xx}^2\left(1-\beta_x^2\right)}{1-\gamma_{xx}^2\left(1-\beta_x^2\right)} v_x + \frac{\gamma_{xx}(v_x/c^2)}{1-\gamma_{xx}^2\left(1-\beta_x^2\right)}\left[\gamma_{yy}v_y\left(u'_y+v_y\right)+\gamma_{zz}v_z\left(u'_z+v_z\right)\right], \tag{16a}$$

$$u'_y = -\frac{(\gamma_{yy}/\gamma_{tt}) - \gamma_{yy}^2\left(1-\beta_y^2\right)}{1-\gamma_{yy}^2\left(1-\beta_y^2\right)} v_y + \frac{\gamma_{yy}(v_y/c^2)}{1-\gamma_{yy}^2\left(1-\beta_y^2\right)}\left[\gamma_{xx}v_x\left(u'_x+v_x\right)+\gamma_{zz}v_z\left(u'_z+v_z\right)\right], \tag{16b}$$

$$u'_z = -\frac{(\gamma_{zz}/\gamma_{tt}) - \gamma_{zz}^2\left(1-\beta_z^2\right)}{1-\gamma_{zz}^2\left(1-\beta_z^2\right)} v_z + \frac{\gamma_{zz}(v_z/c^2)}{1-\gamma_{zz}^2\left(1-\beta_z^2\right)}\left[\gamma_{xx}v_x\left(u'_x+v_x\right)+\gamma_{yy}v_y\left(u'_y+v_y\right)\right], \tag{16c}$$

Since $\gamma_{xx} \ll \gamma_{tt}$, $\gamma_{yy} \ll \gamma_{tt}$, and $\gamma_{zz} \ll \gamma_{tt}$ in Fig. 1, Eqs. (16a), (16b), and 16c) reduce to

$$u'_x = -v_x = -v\cos\phi\sin\theta, \quad u'_y = -v_y = -v\sin\phi\sin\theta, \quad u'_z = -v_z = -v\cos\theta, \tag{17a}$$

$$u_x = v_x = v\cos\phi\sin\theta, \quad u_y = v_y = v\sin\phi\sin\theta, \quad u_z = v_z = v\cos\theta, \tag{17b}$$

which are identical to those in the 6-dimensional spacetime theory [10,11] and their use in finding relativistic mass, energy and Doppler effects is discussed at the end of the manuscript as appendix.

3. Vector transformation and Four Vectors

In this section we lay down the groundwork to study the invariance of relativistic vector quantities between two inertial frames. We momentarily set aside the relativity and focus on the three-dimensional vector transformation [12]. Since coordinate systems are used for convenience; we are free to use the following Figure 2 to define stationary 3-dimensional vectors $\vec{A}'$ and $\vec{A}$ in frames $\bar{\Sigma}'$ and $\bar{\Sigma}$, both of which are initially coincide with a stationary universal inertial frame $\Sigma_\circ$ at $t' = t = t_\circ = 0$.

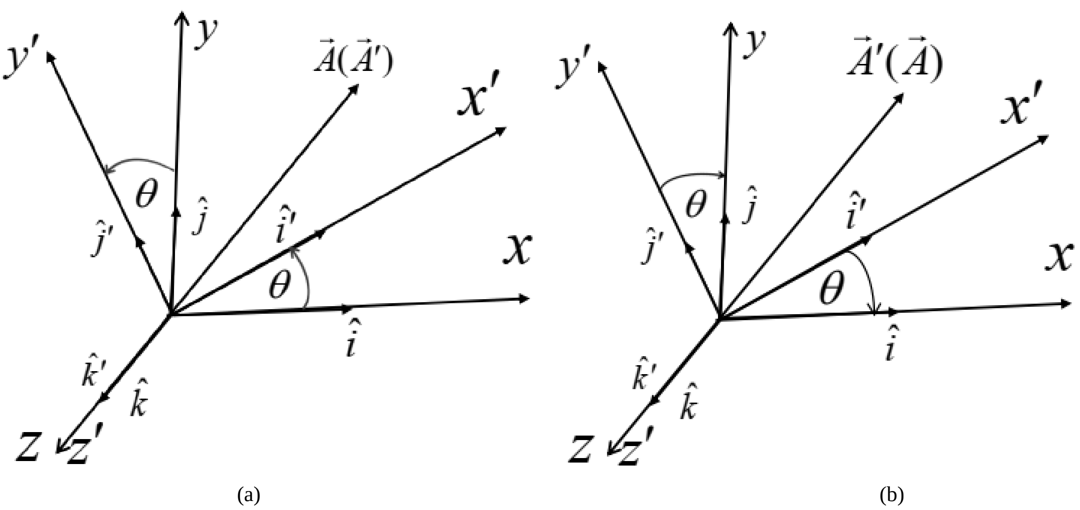

Figure 2. The schematic diagrams of two stationary vectors $\vec{A}'$ and $\vec{A}$ in terms of unit vectors in a counterclockwise rotation in frame $\bar{\Sigma}$ through angle θ into frame $\bar{\Sigma}'$ plane (a) and in clockwise rotation in frame $\bar{\Sigma}'$ through angle θ in clockwise direction into frame $\bar{\Sigma}$ (b) for $0 \le \theta \le \pi/2$

The stationary vectors $\vec{A}'$ and $\vec{A}$ have same length from origin of two frames $\bar{\Sigma}'$ and $\bar{\Sigma}$, written as

$$\vec{A}' = A'_x\hat{i}' + A'_y\hat{j}' + A'_z\hat{k}' = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = \vec{A}, \tag{18}$$

which means that an ordinary vector transformation is identical as if a rotation causes no change of the magnitude of the vector quantity with respect to origin of any massive inertial frame. The unit vectors $(\hat{i}', \hat{j}', \hat{k}')$ and $(\hat{i}, \hat{j}, \hat{k})$ in the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ are defined by using the classical vector transformation [12] and are related to each other according to following equations

$$\begin{pmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}, \tag{19a}$$

$$\begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{pmatrix}, \tag{19b}$$

Consequently, Cartesian components of $\vec{A}'$ and $\vec{A}$ are then written in linear matrix form as [11]

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \tag{20a}$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} \tag{20b}$$

It is noted that three-dimensional rotations can be around any of the three coordinate axes. The 4-dimensional analogue of 3-vector transformation equations given by equations (20a) and (20b) for rotation in counterclockwise and clockwise directions about the z-axis are written as

$$\begin{pmatrix} A'_0 \\ A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_0 \\ A_x \\ A_y \\ A_z \end{pmatrix}, \tag{21a}$$

$$\begin{pmatrix} A_0 \\ A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A'_0 \\ A'_x \\ A'_y \\ A'_z \end{pmatrix} \tag{21b}$$

Here it is noted that, just like 3-dimensional rotations, 4-dimensional rotations can also be around any of 3-coordinate axes. The scalar product of vector $\vec{A}'$ ($\vec{A}$) with itself leads to $\vec{A}'\cdot\vec{A}' = \vec{A}\cdot\vec{A}$, or $|\vec{A}'| = |\vec{A}|$, which states that the magnitude of a vector is Lorentz scalar between frames $\bar{\Sigma}'$ and $\bar{\Sigma}$.

The 4-vector velocities $\vec{A}' = \vec{U}' = (u'_t, u'_x, u'_y, u'_z)$ and $\vec{A} = \vec{U} = (u_t, u_x, u_y, u_z)$ are then defined as

$$\begin{pmatrix} c' \\ u'_x \\ u'_y \\ u'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix}, \tag{22a}$$

$$\begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c' \\ u'_x \\ u'_y \\ u'_z \end{pmatrix}, \tag{22b}$$

where $u'_t = c'$ and $u_t = c$, The scalar product of the 4-velocity vector $\vec{U}'(\vec{U})$ with itself in frame $\bar{\Sigma}'(\bar{\Sigma})$ is equal to the scalar product of $\vec{U}(\vec{U}')$ in frame $\bar{\Sigma}(\bar{\Sigma}')$: $\vec{U}'\cdot\vec{U}' = \vec{U}\cdot\vec{U}$, which allows us to write $|\vec{u}'|^2 - c'^2 = |\vec{u}|^2 - c^2$. Since $|\vec{u}'|^2 = |\vec{u}|^2$, the speed of light is Lorentz invariant ($c' = c$).

The scalar product of momentum 4-vectors $\vec{P}' = (p'_t, p'_x, p'_y, p'_z)$ and $\vec{P} = (p_t, p_x, p_y, p_z)$ with themselves give the same length ($|\vec{P}'| = |\vec{P}|$) from the origins of two frames. The momentum 4-vector transformation is then identified as a rotation if it causes no change in their magnitudes.

4. Conservation of Energy Law and Invariance of Electromagnetic Fields

We will extend our recently proposed 6-dimensional space-time theory [10,11] to write the following linear matrix equations to describe the 4-force vectors $\vec{F}'$ and $\vec{F}$ in terms of each other in frames $\bar{\Sigma}'$ and $\bar{\Sigma}$

$$\begin{pmatrix} F'_0 \\ F'_x \\ F'_y \\ F'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_0 \\ F_x \\ F_y \\ F_z \end{pmatrix}, \tag{23a}$$

$$\begin{pmatrix} F_0 \\ F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F'_0 \\ F'_x \\ F'_y \\ F'_z \end{pmatrix}, \tag{23b}$$

where $F'_0 = d(m_\circ c')/dt' = 0$ and $F_0 = d(m_\circ c)/dt = 0$. (F'_x, F'_y, F'_z) and (F_x, F_y, F_z) are the Cartesian components of the 3-electromagnetic force vectors (Lorentz force) in the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ which, according to 3-dimensional vector transformation equations (21a) and (21b), are

$$\vec{F}' = F'_x\hat{i}' + F'_y\hat{j}' + F'_z\hat{k}' = \left(F_x\cos\theta + F_y\sin\theta\right)\hat{i}' + \left(-F_x\sin\theta + F_y\cos\theta\right)\hat{j}' + F_z\hat{k}' \tag{24a}$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = \left(F'_x\cos\theta - F'_y\sin\theta\right)\hat{i} + \left(F'_x\sin\theta + F'_y\cos\theta\right)\hat{j} + F'_z\hat{k} \tag{24b}$$

Since $F_0 = F_0{}' = 0$ in Eqs. (23a) and (23b), unlike the 4-velocity and 4-momentums, force vector is 3-dimensional as defined by Eqs. (24a) and (24b) in terms of unit vectors $(\hat{i}', \hat{j}', \hat{k}')$ and $(\hat{i}, \hat{j}, \hat{k})$ in frames $\bar{\Sigma}'$ and $\bar{\Sigma}$. The scalar product of $\vec{F}'$ $(\vec{F})$ with itself leads to $\vec{F}'\cdot\vec{F}' = \vec{F}\cdot\vec{F}$, or $|\vec{F}'| = |\vec{F}|$. The rates at which work is done on a particle by Lorentz force in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ are

$$\frac{dE}{dt} = \vec{F}.\vec{u} = F_x u_x + F_y u_y + F_z u_z, \tag{25a}$$

$$\frac{dE'}{dt'} = \vec{F}'.\vec{u}' = F'_x u'_x + F'_y u'_y + F'_z u'_z, \tag{25b}$$

where (F'_x, F'_y, F'_z) and (F_x, F_y, F_z) are Cartesian components of Lorentz force in frames $\bar{\Sigma}'$ and $\bar{\Sigma}$.

$$F'_x = q\left(E'_x + u'_y B'_z - u'_z B'_y\right), \quad F'_y = q\left(E'_y + u'_z B'_x - u'_x B'_z\right), \quad F'_z = q\left(E'_z + u'_x B'_y - u'_y B'_x\right), \tag{26a}$$

$$F_x = q\left(E_x + u_y B_z - u_z B_y\right), \quad F_y = q\left(E_y + u_z B_x - u_x B_z\right), \quad F_z = q\left(E_z + u_x B_y - u_y B_x\right), \tag{26b}$$

where u'_j (u'_k) and u_j (u_k) and are Cartesian components of $\vec{u}'$ and $\vec{u}$ in Eqs. (22a) and (22b).

Considering the massive inertial frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ form a closed system in the stationary spacetime frame Σ_0, the law of conservation of power (or energy) between them is written as

$$\frac{dE'}{dt'} = \frac{dE}{dt} \quad \Rightarrow \quad F'_x u'_x + F'_y u'_y + F'_z u'_z = F_x u_x + F_y u_y + F_z u_z, \tag{27}$$

Using the transformation matrix equations (22a) and (22b) for $\vec{u}'$ and $\vec{u}$, Eq. (27) is written as

$$F'_x u'_x + F'_y u'_y + F'_z u'_z = F_0 c + \left(F_x\cos\theta + F_y\sin\theta\right)u'_x + \left(-F_x\sin\theta + F_y\cos\theta\right)u'_y + F'_z u'_z, \tag{28a}$$

$$F_x u_x + F_y u_y + F_z u_z = F_0' c' + \left(F_x' \cos\theta - F_y' \sin\theta\right) u_x + \left(F_x' \sin\theta + F_y' \cos\theta\right) u_y + F_z' u_z, \tag{28b}$$

where $F_0' = F_0 = 0$. Using Eqs. (26a) and (26b) in Eq. (28a), we can write following equations

$$E_x' + u_y' B_z' - u_z' B_y' = \left(E_x + u_y B_z - u_z B_y\right)\cos\theta + \left(E_y + u_z B_x - u_x B_z\right)\sin\theta, \tag{29a}$$

$$E_y' + u_z' B_x' - u_x' B_z' = -\left(E_x + u_y B_z - u_z B_y\right)\sin\theta + \left(E_y + u_z B_x - u_x B_z\right)\cos\theta, \tag{29b}$$

$$E_z' + u_x' B_y' - u_y' B_x' = E_z + u_x B_y - u_y B_x, \tag{29c}$$

Using Eq. (22a) for u_x', u_y', u_z' in Eqs. (29a), (29b), and (29c), we write following matrix equations

$$\begin{pmatrix} E_x' \\ E_y' \\ E_z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}, \tag{30a}$$

$$\begin{pmatrix} B_x' \\ B_y' \\ B_z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}, \tag{30b}$$

Using Eq. (22b) for u_x, u_y, u_z in Eqs. (29a), (29b), and (29c) for inverse transformation, we write

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x' \\ E_y' \\ E_z' \end{pmatrix}, \tag{30c}$$

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B_x' \\ B_y' \\ B_z' \end{pmatrix}, \tag{30d}$$

which state that electric (or magnetic) field in frame $\bar{\Sigma}'(\bar{\Sigma})$ is composed of electric (or magnetic) field in frame $\bar{\Sigma}(\bar{\Sigma}')$ for spatial rotation of (x, y) plane about z-axis.

The scalar and vector products of electric and magnetic fields is essential in proving Lorentz invariance of electromagnetic energy, Poynting vector, current continuity equation, and densities of electromagnetic field energy and momentum. Using Eqs. (30a) -(30d) one writes

$$\begin{aligned} \vec{E}' \cdot \vec{B}' &= E_x' B_x' + E_y' B_y' + E_z' B_z' = \left(E_x \cos\theta + E_y \sin\theta\right)\left(B_x \cos\theta + B_y \sin\theta\right) \\ &+ \left(-E_x \sin\theta + E_y \cos\theta\right)\left(-B_x \sin\theta + B_y \cos\theta\right) + \left(E_z B_z\right) = E_x B_x + E_y B_y + E_z B_z = \vec{E} \cdot \vec{B}, \end{aligned} \tag{31a}$$

$$\begin{aligned} \vec{E} \cdot \vec{B} &= E_x B_x + E_y B_y + E_z B_z = \left(E_x' \cos\theta - E_y' \sin\theta\right)\left(B_x' \cos\theta - B_y' \sin\theta\right) \\ &+ \left(E_x' \sin\theta + E_y' \cos\theta\right)\left(B_x' \sin\theta + B_y' \cos\theta\right) + \left(E_z' B_z'\right) = E_x' B_x' + E_y' B_y' + E_z' B_z' = \vec{E}' \cdot \vec{B}', \end{aligned} \tag{31b}$$

which suggests that $\vec{E}' \cdot \vec{E}' = \vec{E} \cdot \vec{E}$ and $\vec{B}' \cdot \vec{B}' = \vec{B} \cdot \vec{B}$, so that $E' = E$ and $B' = B$ (Lorentz scalar invariants). The vector products of $\vec{E}'$ and $\vec{B}'$ in frame Σ' and of $\vec{E}$ and $\vec{B}$ frames Σ' and Σ are

$$\begin{aligned} \vec{E}' \times \vec{B}' &= \left(E_y' B_z' - B_y' E_z'\right)\hat{i}' + \left(B_x' E_z' - E_x' B_z'\right)\hat{j}' + \left(E_x' B_y' - B_x' E_y'\right)\hat{k}' \\ &= \left(\left(E_y B_z - E_z B_y\right)\cos\theta + \left(E_z B_x - E_x B_z\right)\sin\theta\right)\hat{i}' \\ &+ \left(\left(E_z B_x - E_x B_z\right)\cos\theta + \left(E_z B_y - E_y B_z\right)\sin\theta\right)\hat{j}' + \left(E_x B_y - E_y B_x\right)\hat{k}' = \vec{E} \times \vec{B}, \end{aligned} \tag{32a}$$

$$\begin{aligned}\vec{E}\times\vec{B} &= \left(E_yB_z - E_zB_y\right)\hat{i} + \left(E_zB_x - E_xB_z\right)\hat{j} + \left(E_xB_y - E_yB_x\right)\hat{k} \\ &= \left(\left(E'_yB'_z - E'_zB'_y\right)\cos\theta + \left(E'_xB'_z - E'_zB'_x\right)\sin\theta\right)\hat{i} \\ &\quad + \left(\left(E'_zB'_x - E'_xB'_z\right)\cos\theta - \left(E'_zB'_y - E'_yB'_z\right)\sin\theta\right)j + \left(E_xB_y - E_yB_x\right)\hat{k} = \vec{E}'\times\vec{B}',\end{aligned} \tag{32b}$$

which states that $\vec{E}'\times\vec{B}' = \vec{E}\times\vec{B}$ is Lorentz invariant vector between the frames Σ' and Σ.

5. Conservation of Energy and Faraday Tensor in Field Transformation

We will extend 3-dimensional Lorentz force $\vec{F} = Q_e(\vec{E} + \vec{u}\times\vec{B})$ to 4-dimensions as covariant and contravariant Lorentz tensors $F_\alpha = Q_eF_{\alpha\beta}U^\beta$ and $F^\alpha = Q_eF^{\alpha\beta}U_\beta$, respectively. Here Q_e is the static electric charge, $F_{\alpha\beta}(F^{\alpha\beta})$ is the covariant contravariant antisymmetric second rank tensor, also known as Faraday tensor. $U_\beta = Tr(u_t, -u_x, -u_y, -u_z)$ and $U^\beta = Tr(u_t, u_x, u_y, u_z)$ are covariant and contravariant 4-vector velocities. Considering the frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ form a closed system in the stationary frame $\sum_0$, the law of conservation of power (energy) equation (27) can be written as

$$\frac{dE'}{dt'} = \frac{dE}{dt} \quad \Rightarrow \quad F'_\alpha U'^\alpha = F_\alpha U^\alpha \qquad \Rightarrow \qquad Q_eF'_{\alpha\beta}U'^\beta U'^\alpha = Q_eF_{\alpha\beta}U^\beta U^\alpha, \tag{33}$$

where $U_\alpha = U_\beta$ and $U^\alpha = U^\beta$. Using $U'^\alpha = R(\theta)U^\alpha$ and $U^\beta = \tilde{R}(\theta)U'^\beta$ in Eq. (33) we write the following rule to transform $F_{\alpha\beta}$ in $\bar{\Sigma}$ into $\bar{\Sigma}'$ for a counter-clockwise rotation about z-axis.

$$F'_{\alpha\beta} = R(\theta)F_{\alpha\beta}\tilde{R}(\theta), \tag{34}$$

where $\tilde{R}(\theta) = R^{-1}(\theta)$ is the transpose (inverse) of rotation matrix $R(\theta)$.

In tensor analysis, the covariant (contravariant) tensor $F_{\alpha\beta}(F^{\alpha\beta})$ is defined as vector product of any two vectors $\vec{a}$ and $\vec{b}$ is another vector $\vec{c} = \vec{a}\times\vec{b} = \left(a_ib_j - a_jb_i\right)\hat{n}_k$ and considered as second rank antisymmetric tensor [4], with $i, j \neq k$. Covariant and contravariant 4-Lorentz forces as 4-tensors are defined as $F_\alpha = Q_eF_{\alpha\beta}U^\beta$ and $F^\alpha = Q_eF^{\alpha\beta}U_\beta$ in frame $\bar{\Sigma}$, written as

$$\begin{pmatrix} F_0 \\ F_x \\ F_y \\ F_z \end{pmatrix} = Q_e \begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} u_t \\ u_x \\ u_y \\ u_z \end{pmatrix}, \tag{35a}$$

$$\begin{pmatrix} F^0 \\ F^x \\ F^y \\ F^z \end{pmatrix} = Q_e \begin{pmatrix} F^{00} & F^{01} & F^{01} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} \begin{pmatrix} u_t \\ -u_x \\ -u_y \\ -u_z \end{pmatrix}, \tag{35b}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is second rank antisymmetric covariant field tensor, also called Faraday tensors, with $A_\alpha(A'_\alpha)$ being any arbitrary 4-vector in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$. $F_{\alpha\beta}$ are defined as

$$F_{00} = F_{11} = F_{22} = F_{33} = 0, F_{10} = -F_{01},\ F_{20} = -F_{02},\ F_{30} = -F_{03}, F_{21} = -F_{12},\ F_{31} = -F_{13}, F_{32} = -F_{23}.$$

Similarly, one can define the contravariant Faraday tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$.

Dual Faraday tensor transformation between frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ is defined like Faraday tensor transformation equation (33). To do that we first introduce a "fictitious" magnetic charge Q_m [13] and define "fictitious" dual Lorentz force $\vec{F}_m = Q_m(c\vec{B} - \vec{u}\times\vec{E}/c)$ by using electromagnetic duality $\vec{E}/c \rightleftarrows \vec{B}$ and $\vec{B} \rightleftarrows -\vec{E}/c$ in conventional Lorentz force

$\vec{F}_e = Q_e(\vec{E} + \vec{u} \times \vec{B})$. Here u is the velocity of moving electrical charge. We then define 4- dual covariant and contravariant Lorentz tensors in frame $\bar{\Sigma}$ as $F_\alpha = Q_m G_{\alpha\beta} U^\beta$, $F^\alpha = Q_m G^{\alpha\beta} U_\beta$. Using $U'^\alpha = R(\theta) U^\alpha$ and $U^\beta = \tilde{R}(\theta) U'^\beta$ we can write the following rule to transform $G_{\alpha\beta}$ in $\bar{\Sigma}$ into $\bar{\Sigma}'$ for a counter-clockwise rotation about z-axis.

$$G'_{\alpha\beta} = R(\theta) G_{\alpha\beta} \tilde{R}(\theta), \tag{36}$$

where $\tilde{R}(\theta) = R^{-1}(\theta)$ is the transpose (inverse) of rotation matrix $R(\theta)$. Covariant and contravariant fictitious 4-vector dual Lorentz forces F_α and F^α in frame $\bar{\Sigma}$ are then written as

$$\begin{pmatrix} F_0 \\ F_x \\ F_y \\ F_z \end{pmatrix} = Q_m \begin{pmatrix} G_{00} & G_{01} & G_{02} & G_{03} \\ G_{10} & G_{11} & G_{12} & G_{13} \\ G_{20} & G_{21} & G_{22} & G_{23} \\ G_{30} & G_{31} & G_{32} & G_{33} \end{pmatrix} \begin{pmatrix} u_t \\ u_x \\ u_y \\ u_z \end{pmatrix}, \tag{37a}$$

$$\begin{pmatrix} F'_0 \\ F'_x \\ F'_y \\ F'_z \end{pmatrix} = Q_m \begin{pmatrix} G^{00} & G^{01} & G^{01} & G^{03} \\ G^{10} & G^{11} & G^{12} & G^{13} \\ G^{20} & G^{21} & G^{22} & G^{23} \\ G^{30} & G^{31} & G^{32} & G^{33} \end{pmatrix} \begin{pmatrix} u_t \\ -u_x \\ -u_y \\ -u_z \end{pmatrix}, \tag{37b}$$

where $G_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is antisymmetric second rank covariant field tensor. Similarly, one can define contravariant dual tensor $G^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$. Components of the covariant (contravariant) dual tensors $G_{\alpha\beta}(G^{\alpha\beta})$ are then obtained by using $\vec{E}/c \rightleftarrows \vec{B}$ and $\vec{B} \rightleftarrows -\vec{E}/c$ in $F_{\alpha\beta}(F^{\alpha\beta})$.

To eliminate the unnoticed error made in the classical use of Faraday tensor in field transformation [4] we use the generalized 4-dimensional Minkowski spacetime wherein 4-vector space coordinates are $x_\alpha = x_\alpha(ict, -x, -y, -z)$ and $x'_\alpha = x'_\alpha(ic't', -x', -y', -z')$ in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$.

Using the covariant 4-vector potentials $A_\alpha = \left\{V/c, -\vec{A}/c\right\}$, $A'_\alpha = \left\{V'/c', -\vec{A}'/c'\right\}$ and multiplying $\vec{E}$ and $\vec{E}'$ with (i/c) and (i/c') the electric and magnetic fields in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ are written as

$$\frac{i}{c}\vec{E} = -\frac{i}{c}\vec{\nabla}V - \frac{i}{c}\frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}; \qquad \frac{i}{c'}\vec{E}' = -\frac{i}{c'}\vec{\nabla}'V' - \frac{i}{c}\frac{\partial \vec{A}'}{\partial t'}, \quad \vec{B}' = \nabla' \times \vec{A}', \tag{38}$$

We introduce covariant Faraday tensor $F_{\alpha\beta}$ with the following Cartesian components in frame $\bar{\Sigma}$

$$i\frac{E_x}{c} = -\frac{i}{c}\left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t}\right) = F_{01} = -F_{10}, \qquad B_x = \left(\vec{\nabla} \times \vec{A}\right)_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = F_{32} = -F_{23}, \tag{39a}$$

$$i\frac{E_y}{c} = -\frac{i}{c}\left(\frac{\partial V}{\partial y} + \frac{\partial A_y}{\partial t}\right) = F_{02} = -F_{20}, \qquad B_y = \left(\vec{\nabla} \times \vec{A}\right)_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_y}{\partial x} = F_{31} = -F_{13}, \tag{39b}$$

$$i\frac{E_z}{c} = -\frac{i}{c}\left(\frac{\partial V}{\partial z} + \frac{\partial A_z}{\partial t}\right) = F_{03} = -F_{30}, \qquad B_z = \left(\vec{\nabla} \times \vec{A}\right)_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = F_{21} = -F_{12}, \tag{39c}$$

Covariant and contravariant Faraday tensors $F_{\alpha\beta}$ and $F^{\alpha\beta}$, and their duals $G_{\alpha\beta}$ and $G^{\alpha\beta}$ in frame $\bar{\Sigma}$ are

$$F_{\alpha\beta} = \begin{pmatrix} 0 & iE_x/c & iE_y/c & iE_z/c \\ -iE_x/c & 0 & -B_z & B_y \\ -iE_y/c & B_z & 0 & -B_x \\ -iE_z/c & -B_y & B_x & 0 \end{pmatrix}, \tag{40a}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -iE_x/c & -iE_y/c & -iE_z/c \\ iE_x/c & 0 & -B_z & B_y \\ iE_y/c & B_z & 0 & -B_x \\ iE_z/c & -B_y & B_x & 0 \end{pmatrix}, \tag{40b}$$

$$G_{\alpha\beta} = \begin{pmatrix} 0 & iB_x & iB_y & iB_z \\ -iB_x & 0 & E_z/c & -E_y/c \\ -iB_y & -E_z/c & 0 & E_x/c \\ -iB_z & E_y/c & -E_x/c & 0 \end{pmatrix}, \tag{40c}$$

$$G^{\alpha\beta} = \begin{pmatrix} 0 & -iB_x & -iB_y & -iB_z \\ iB_x & 0 & E_z/c & -E_y/c \\ iB_y & -E_z/c & 0 & E_x/c \\ iB_z & E_y/c & -E_x/c & 0 \end{pmatrix}, \tag{40d}$$

where the components of the dual Faraday tensors $G_{\alpha\beta}$ and $G^{\alpha\beta}$ are constructed by using the so called the electromagnetic duality $\vec{E}/c \rightleftarrows \vec{B}$ and $\vec{B} \rightleftarrows -\vec{E}/c$ in Faraday tensors $F_{\alpha\beta}$ and $F^{\alpha\beta}$. Using equations (40a), (40b), (40c), and (40d) for the covariant and contravariant Faraday tensors ($F_{\alpha\beta}$ and $F^{\alpha\beta}$) and their duals ($G_{\alpha\beta}$ and $G^{\alpha\beta}$), one can write the following matrix expressions for the products $F_{\alpha\beta}F^{\alpha\beta}$ and of their duals $G_{\alpha\beta}G^{\alpha\beta}$ in frame $\bar{\Sigma}$

$$F_{\alpha\beta}F^{\alpha\beta} = \begin{pmatrix} \frac{i^2}{c^2}\vec{E}^2 & \frac{i}{c}\mu_\circ S_x & \frac{i}{c}\mu_\circ S_y & \frac{i}{c}\mu_\circ S_z \\ -\frac{i}{c}\mu_\circ S_x & \left(\frac{i^2}{c^2}E_x^2 - B_y^2 - B_z^2\right) & \left(\frac{i^2}{c^2}E_xE_y + B_xB_y\right) & \left(\frac{i^2}{c^2}E_xE_z + B_xB_z\right) \\ -\frac{i}{c}\mu_\circ S_y & \left(\frac{i^2}{c^2}E_yE_x + B_yB_x\right) & \left(\frac{i^2}{c^2}E_y^2 - B_z^2 - B_x^2\right) & \left(\frac{i^2}{c^2}E_yE_z + B_yB_z\right) \\ -\frac{i}{c}\mu_\circ S_z & \left(\frac{i^2}{c^2}E_zE_x + B_zB_x\right) & \left(\frac{i^2}{c^2}E_zE_y + B_zB_y\right) & \left(\frac{i^2}{c^2}E_z^2 - B_x^2 - B_y^2\right) \end{pmatrix}, \tag{41}$$

$$G_{\alpha\beta}G^{\alpha\beta} = \begin{pmatrix} i^2\vec{B}^2 & \frac{i}{c}\mu_\circ S_x & \frac{i}{c}\mu_\circ S_y & \frac{i}{c}\mu_\circ S_z \\ -\frac{i}{c}\mu_\circ S_x & \left(i^2B_x^2 - \frac{1}{c^2}E_y^2 - \frac{1}{c^2}E_z^2\right) & \left(\frac{1}{c^2}E_xE_y + i^2B_xB_y\right) & \left(\frac{1}{c^2}E_xE_z + i^2B_xB_z\right) \\ -\frac{i}{c}\mu_\circ S_y & \left(\frac{1}{c^2}E_yE_x + i^2B_yB_x\right) & \left(i^2B_y^2 - \frac{1}{c^2}E_z^2 - \frac{1}{c^2}E_x^2\right) & \left(\frac{1}{c^2}E_zE_y + i^2B_yB_x\right) \\ -\frac{i}{c}\mu_\circ S_z & \left(\frac{1}{c^2}E_zE_x + i^2B_zB_x\right) & \left(\frac{1}{c^2}E_zE_y + i^2B_zB_y\right) & \left(i^2B_z^2 - \frac{1}{c^2}E_x^2 - \frac{1}{c^2}E_y^2\right) \end{pmatrix}, \tag{42}$$

where $S_x = (E_yB_z - E_zB_y)/\mu_\circ$, $S_y = (E_zB_x - E_xB_z)/\mu_\circ$, and $S_z = (E_xB_y - E_yB_x)/\mu_\circ$ are the *x*, *y*, and *z* components of the Poynting vector. Traces of $F_{\alpha\beta}F^{\alpha\beta}$ and of $G_{\alpha\beta}G^{\alpha\beta}$ are nonzero:

$$Tr\left(F_{\alpha\beta}F^{\alpha\beta}\right) = -4\mu_\circ u_{em} \neq 0 \tag{43a}$$

$$Tr(G_{\alpha\beta}G^{\alpha\beta}) = -4\mu_\circ u_{em} \neq 0, \tag{43b}$$

which is physically realistic because the electromagnetic waves transfer energy and momentum in free space. Traces of product of covariant and contravariant Faraday tensors with their duals are

$$Tr\left(F_{\alpha\beta}G_{\alpha\beta}\right)=Tr\left(F^{\alpha\beta}G^{\alpha\beta}\right)=\frac{4}{c}\left(\vec{E}\cdot\vec{B}\right), \tag{44a}$$

$$Tr\left(F_{\alpha\beta}G^{\alpha\beta}\right)=Tr\left(F^{\alpha\beta}G_{\alpha\beta}\right)=\frac{-2}{c}\left(\vec{E}\cdot\vec{B}\right)+\frac{2}{c}\left(\vec{E}\cdot\vec{B}\right)=0, \tag{44b}$$

Equation (44a) suggests that $F_{\alpha\beta}\,(F^{\alpha\beta})$ and $G_{\alpha\beta}(G^{\alpha\beta})$ are orthogonal only when $\vec{E}\cdot\vec{B}$ is zero.

Equation. (44b) suggests that $F_{\alpha\beta}\,(F^{\alpha\beta})$ and $G^{\alpha\beta}\,(G_{\alpha\beta})$ are always orthogonal in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$.

6. Faraday Tensor in Electromagnetic Field Transformation

The incremental displacement of a coordinate system relative to its initial position is known to be composed of a translation as well as a rotation. While the components of 4-vector quantities transform according to Eq. (21a) and (21b), the spatial translation has no effect on them. One can use Faraday tensors and their duals determine the components of electric and magnetic fields in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ by using:

(i) **Lorentz boost** along the direction of motion, which is appropriate transformation for relative motion of two observers, and
(ii) **Spatial rotation**, which relates spacetime coordinates of two observers in two frames which are subject to planar rotation with respect to each other about a fixed axis.

6.1. Faraday Tensor and Field Transformation with Lorentz Boost

We use $F'_{\alpha\beta,z}=L_zF_{\alpha\beta}\tilde{L}_z$ $F'_{\alpha\beta,y}=L_yF_{\alpha\beta}\tilde{L}_y$, and $F'_{\alpha\beta,x}=L_xF_{\alpha\beta}\tilde{L}_x$ transformation rules to transform electromagnetic fields between frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ for Lorentz boost along x, y, and z-axes, which are written as

$$L_z=\begin{pmatrix}\gamma & i\gamma\beta & 0 & 0\\ -i\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\quad L_y=\begin{pmatrix}\gamma & 0 & i\gamma\beta & 0\\ 0 & 1 & 0 & 0\\ -i\gamma\beta & 0 & \gamma & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\quad L_x=\begin{pmatrix}\gamma & 0 & 0 & i\gamma\beta\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -i\gamma\beta & 0 & & \gamma\end{pmatrix}, \tag{45}$$

where $\gamma=\cosh\xi$, $\gamma\beta=\sinh\xi$ with $0\le\beta\le 1$ and $0\le\gamma\le\infty$. Here $\xi=\tanh^{-1}(\beta)$ is called the boost parameter [4]. For a Lorentz boost along x-axis, the transformation $F'_{\alpha\beta,z}=L_zF_{\alpha\beta}\tilde{L}_z$ allows us to obtain the Cartesian components of Faraday tensor in frame $\bar{\Sigma}'$ in terms of those in frame $\bar{\Sigma}$

$$\begin{pmatrix}0 & E'_x/c & E'_y/c & E'_z/c\\ -E'_x/c & 0 & -B'_z & B'_y\\ -E'_y/c & B'_z & 0 & -B'_x\\ -E'_z/c & -B'_y & B'_x & 0\end{pmatrix}=\begin{pmatrix}\gamma & i\gamma\beta & 0 & 0\\ -i\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & E_x/c & E_y/c & E_z/c\\ -E_x/c & 0 & -B_z & B_y\\ -E_y/c & B_z & 0 & -B_x\\ -E_z/c & -B_y & B_x & 0\end{pmatrix}\begin{pmatrix}\gamma & -i\gamma\beta & 0 & 0\\ i\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{46}$$

which yields the covariant Faraday tensor in frame $\bar{\Sigma}'$ in terms of those in frame $\bar{\Sigma}$. Transformation $F'_{\alpha\beta,z}=L_zF_{\alpha\beta}\tilde{L}_z$ and $F_{\alpha\beta,z}=\tilde{L}_zF'_{\alpha\beta}L_z$ yield electric and magnetic field components in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$

$$\begin{pmatrix}E'_x\\ E'_y\\ E'_z\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}E_x\\ \gamma\left(E_y+\beta B_z\right)\\ \gamma\left(E_z-\beta B_y\right)\end{pmatrix}, \tag{47a}$$

$$\begin{pmatrix}B'_x\\ B'_y\\ B'_z\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}B_x\\ \gamma\left(B_y-(v/c^2)E_z\right)\\ \gamma\left(B_z+(v/c^2)E_y\right)\end{pmatrix} \tag{47b}$$

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E'_x \\ \gamma\left(E'_y + vB'_z\right) \\ \gamma\left(E'_z - vB'_y\right) \end{pmatrix}, \tag{47c}$$

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B'_x \\ \gamma\left(B'_y + (v/c^2)E'_z\right) \\ \gamma\left(B'_z - (v/c^2)E'_y\right) \end{pmatrix} \tag{47d}$$

which are the same as Eqs. (4a) and (4b) and yield invariant scalar product $(\vec{E}'\cdot\vec{B}' = \vec{E}\cdot\vec{B})$ and non-invariant vector product $(\vec{E}'\times\vec{B}' \neq \vec{E}\times\vec{B})$ between two frames under Lorentz transformation.

In the next step, we introduce a circular boost, rather than hyperbolic one defined in Eq. (45), for motion along x, y, and z-axes in frame $\bar{\Sigma}$, $F'_{\alpha\beta,x}(\theta) = L_x(\theta)F_{\alpha\beta}\tilde{L}_x(\theta)$ to find $F'_{\alpha\beta}(F'^{\alpha\beta})$, which are

$$L_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_x(\theta) = \begin{pmatrix} \cos\theta & 0 & 0 & \sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta & 0 & & \cos\theta \end{pmatrix}, \tag{48}$$

For a circular boost along x-axis, the components of covariant tensor $F'_{\alpha\beta}$ in frame $\bar{\Sigma}'$ are obtained by using the transformation rule $F'_{\alpha\beta,z}(\theta) = L_z(\theta)F_{\alpha\beta}\tilde{L}_z(\theta)$, which is explicitly written as

$$\begin{aligned} &\begin{pmatrix} 0 & iE'_x/c' & iE'_y/c' & iE'_z/c' \\ -iE'_x/c' & 0 & -B'_z & B'_y \\ -iE'_y/c' & B'_z & 0 & -B'_x \\ -iE'_z/c' & -B'_y & B'_x & 0 \end{pmatrix} = \\ &\begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & iE_x/c & iE_y/c & iE_z/c \\ -iE_x/c & 0 & -B_z & B_y \\ -iE_y/c & B_z & 0 & -B_x \\ -iE_z/c & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{49}$$

$$\begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y\cos\theta - B_z\sin\theta \\ E_z\cos\theta + B_y\sin\theta \end{pmatrix}, \tag{50a}$$

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B_x \\ B_y\cos\theta - E_z\sin\theta \\ B_z\cos\theta + E_y\sin\theta \end{pmatrix}, \tag{50b}$$

with inverse transformation rule $F_{\alpha\beta,z}(\theta) = \tilde{L}_z(\theta)F'_{\alpha\beta}L_z(\theta)$ one writes $F_{\alpha\beta}$ in frame Σ and obtain

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E'_x \\ E'_y\cos\theta + B'_z\sin\theta \\ E'_z\cos\theta - B'_y\sin\theta \end{pmatrix}, \tag{50c}$$

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B'_x \\ B'_y \cos\theta + E'_z \sin\theta \\ B'_z \cos\theta - E'_y \sin\theta \end{pmatrix}, \tag{50d}$$

which yield Lorentz invariant scalar product of electric and magnetic fields $(\vec{E}'\cdot\vec{B}'=\vec{E}\cdot\vec{B})$ between frames $\bar{\Sigma}$ and $\bar{\Sigma}'$. The vector products of electric and magnetic in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ fields are

$$\begin{aligned} \vec{E}'\times\vec{B}' &= \Big[\big(E_yB_z - E_zB_y\big)\cos^2\theta + \big(E_zB_y - E_yB_z\big)\sin^2\theta + \big(E_y^2 + E_z^2 + B_y^2 + B_z^2\big)\sin\theta\cos\theta\Big]\hat{i}' \\ &+ \Big[\big(E_zB_x - E_xB_z\big)\cos\theta + \big(B_xB_y - E_xE_y\big)\sin\theta\Big]\hat{j}' + \Big[\big(E_xB_y - E_yB_x\big)\cos\theta + \big(B_xB_z - E_xE_z\big)\sin\theta\Big]\hat{k}', \end{aligned} \tag{51a}$$

$$\begin{aligned} \vec{E}\times\vec{B} &= \Big[\big(E'_yB'_z - E'_zB'_y\big)\cos^2\theta + \big(E'_zB'_y - E'_yB'_z\big)\sin^2\theta + \big(-E'^2_y - E'^2_z + B'^2_y + B'^2_z\big)\sin\theta\cos\theta\Big]\hat{i}' \\ &+ \Big[\big(E'_zB'_x - E'_xB'_z\big)\cos\theta + \big(E'_xE'_y - B'_xB'_y\big)\sin\theta\Big]\hat{j}' + \Big[\big(E'_xB'_y - E'_yB'_x\big)\cos\theta + \big(E'_xE'_z - B'_xB'_z\big)\sin\theta\Big]\hat{k}', \end{aligned} \tag{51b}$$

When $\theta = 0^\circ$ unit vectors are equivalent in both frames ($\hat{i}'=\hat{i}$, $\hat{j}'=\hat{j}$, and $\hat{k}'=\hat{k}$). Eqs. (51a) and (51b) yield $\vec{E}'\times\vec{B}'=\vec{E}\times\vec{B}$ and $\vec{E}\times\vec{B}=\vec{E}'\times\vec{B}'$, contrary to the hyperbolic boosts in Eq. (45).

6.2. Faraday Tensor and Field Transformation with Spatial Rotations

It is possible to combine boosts with rotations to relate the electromagnetic fields to each other between the frames $\bar{\Sigma}$ and $\bar{\Sigma}'$, by using counterclockwise rotations of (x, y), (z, x), and (y, z) planes about z, y and x-axes, which are given by the following expressions

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}, \quad R_y(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad R_z(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{52}$$

Replacing θ with $-\theta$ one writes expressions for the clockwise rotation about x, y, and z-axes. As an example, the transformation $F'_{\alpha\beta,z} = R_z(\theta)F_{\alpha\beta}\tilde{R}_z(\theta)$ allows us to determine the components of covariant Faraday tensor $F'_{\alpha\beta}$ in frame $\bar{\Sigma}'$ in terms of the components of covariant $F_{\alpha\beta}$ in frame $\bar{\Sigma}$

$$\begin{aligned} &\begin{pmatrix} 0 & iE'_x/c' & iE'_y/c' & iE'_z/c' \\ -iE'_x/c' & 0 & -B'_z & B'_y \\ -iE'_y/c' & B'_z & 0 & -B'_x \\ -iE'_z/c' & -B'_y & B'_x & 0 \end{pmatrix} = \\ &\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & iE_x/c & iE_y/c & iE_z/c \\ -iE_x/c & 0 & -B_z & B_y \\ -iE_y/c & B_z & 0 & -B_x \\ -iE_z/c & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{53}$$

with inverse transformation $F_{\alpha\beta,z} = \tilde{R}_z(\theta)F'_{\alpha\beta}R_z(\theta)$ one writes similar equation for $F_{\alpha\beta}$ in frame Σ. Matching both sides of Eq. (53) and of its inverse, one writes following matrix equations

$$\begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}, \tag{54a}$$

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}, \tag{54b}$$

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix}, \tag{54c}$$

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix}, \tag{54d}$$

which are identical to Eqs. (30a), (30b) and (30c), and (30d) and yield invariant scalar and vector products of electric and magnetic fields in frames Σ' and Σ,

$$\vec{E}'\cdot\vec{B}' = \vec{E}\cdot\vec{B}, \qquad \rightleftarrows \qquad \vec{E}\cdot\vec{B} = \vec{E}'\cdot\vec{B}', \tag{55a}$$

$$\vec{E}'\times\vec{B}' = \vec{E}\times\vec{B}, \qquad \rightleftarrows \qquad \vec{E}\times\vec{B} = \vec{E}'\times\vec{B}', \tag{55b}$$

which are both invariants between frames Σ' and Σ.

7. Faraday Tensor and Invariance of Maxwell's Equations

Using $J_\alpha = (ic\rho, -\vec{J})$ in $\partial^\alpha F_{\alpha\beta} - \mu_\circ J^\alpha = 0$ and $\partial^\alpha G_{\alpha\beta} = 0$ with $\beta = 0,1,2,3$ for $\alpha(=0,1,2,3)$ we write

$$\partial^0 F_{0\beta} - \mu_\circ J_0 = \frac{i}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) - \frac{i}{c}\left(\mu_\circ c^2 \rho\right) = 0 \qquad \Rightarrow \qquad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_\circ}, \tag{56a}$$

$$\left.\begin{aligned} \partial^1 F_{1\beta} - \mu_\circ J_1 &= -\frac{i}{c}\frac{\partial E_x}{\partial(ict)} + \mu_\circ\left(\vec{\nabla}\times\vec{H}\right)_x - \mu_\circ J_x = 0 \\ \partial^2 F_{2\beta} - \mu_\circ J_2 &= -\frac{i}{c}\frac{\partial E_y}{\partial(ict)} + \mu_\circ\left(\vec{\nabla}\times\vec{H}\right)_y - \mu_\circ J_y = 0 \\ \partial^3 F_{3\beta} - \mu_\circ J_3 &= -\frac{i}{c}\frac{\partial E_z}{\partial(ict)} + \mu_\circ\left(\vec{\nabla}\times\vec{H}\right)_z - \mu_\circ J_z = 0 \end{aligned}\right\} \quad \Rightarrow \quad \vec{\nabla}\times\vec{H} = \vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}, \tag{56b}$$

$$\partial^0 G_{0\beta} = i\mu_\circ\left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right) = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{H} = 0, \tag{57a}$$

$$\left.\begin{aligned} \partial^1 G_{1\beta} &= i\mu_\circ\frac{\partial H_x}{\partial(ict)} + \frac{1}{c}\left(\vec{\nabla}\times\vec{E}\right)_x = 0 \\ \partial^2 G_{2\beta} &= i\mu_\circ\frac{\partial H_y}{\partial(ict)} + \frac{1}{c}\left(\vec{\nabla}\times\vec{E}\right)_y = 0 \\ \partial^3 G_{3\beta} &= i\mu_\circ\frac{\partial H_z}{\partial(ict)} + \frac{1}{c}\left(\vec{\nabla}\times\vec{E}\right)_z = 0 \end{aligned}\right\} \quad \Rightarrow \quad \vec{\nabla}\times\vec{E} = -\mu_\circ\frac{\partial\vec{H}}{\partial t}, \tag{57b}$$

Subtracting Faraday's law of induction from Ampere-Maxwell's equation and applying the divergence to the resultant equation we can write the following equation

$$\vec{\nabla}\cdot\vec{\nabla}\times(\vec{H}-\vec{E}) = \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}\left(\varepsilon_\circ\vec{\nabla}\cdot\vec{E} + \mu_\circ\vec{\nabla}\cdot\vec{H}\right), \tag{58}$$

Since $\vec{\nabla}\cdot\vec{\nabla}\times(\vec{H}-\vec{E}) = 0$, $\vec{\nabla}\cdot\vec{E} = \rho_e/\varepsilon_\circ$ and $\vec{\nabla}\cdot\vec{H} = 0$, equation (58) results in the conventional current continuity equation written as

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0, \tag{59}$$

where $\vec{J}$ and ρ are the total current and charge densities, respectively. In the following subsections we will prove that inhomogeneous and homogeneous Maxwell equations (56) and (57) are Lorentz invariant between two massive inertial frames. Recall the following expressions for the flux of a vector field through a spherical closed surface in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$

$$\Phi=\oint_S \vec{V}(\vec{r}).d\vec{A}=\oint_V \vec{\nabla}\cdot\vec{V}(\vec{r})dv=\frac{1}{\varepsilon_0}\oint_V \rho dv, \qquad \Phi'=\oint_{S'} \vec{V}'(\vec{r}).d\vec{A}'=\oint_{V'} \vec{\nabla}'\cdot\vec{E}'dv'=\frac{1}{\varepsilon_0'}\oint_{V'} \rho' dv', \tag{60}$$

Since any function (e.g., magnitude of electric and magnetic fields) is continuous at any point in space [16] in both frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ $(\Phi=\Phi')$, we write the following chain rules of differentiation

$$\frac{\partial\Phi}{\partial x_i}=\frac{\partial\Phi'}{\partial x_i}=\frac{\partial x_i'}{\partial x_i}\frac{\partial\Phi'}{\partial x_i'}+\frac{\partial t'}{\partial x_i}\frac{\partial\Phi'}{\partial t'}=\frac{\partial x_i'}{\partial x_i}\frac{\partial\Phi'}{\partial x_i'}+\frac{\partial t'}{\partial x_i}\frac{\partial x_i}{\partial t'}\frac{\partial x_i'}{\partial x_i}\frac{\partial\Phi'}{\partial x_i'}=\left(1-\beta_{x_i}'^2\right)\frac{\partial\Phi'}{\partial x_i'}, \tag{61}$$

$$\frac{\partial\Phi}{\partial t}=\frac{\partial\Phi'}{\partial t}=\frac{\partial\Phi'}{\partial t'}\frac{\partial t'}{\partial t}+\frac{\partial\Phi'}{\partial t'}\frac{\partial t'}{\partial x}\frac{\partial x}{\partial t'}\frac{\partial t'}{\partial t}+\frac{\partial\Phi'}{\partial t'}\frac{\partial t'}{\partial y}\frac{\partial y}{\partial t'}\frac{\partial t'}{\partial t}+\frac{\partial\Phi'}{\partial t'}\frac{\partial t'}{\partial z}\frac{\partial z}{\partial t'}\frac{\partial t'}{\partial t}=\left(1-\beta'^2\right)\frac{\partial\Phi'}{\partial t'}, \tag{62}$$

7.1. Gauss Law of Electrostatics

Since electric field wave function is continuous at any point in space, taking $\Phi=E$ and $\Phi'=E'$, and applying chain rule in Eq. (61) we write the charge density

$$\rho=\varepsilon_\circ\vec{\nabla}.\vec{E}=\varepsilon_\circ\gamma_{xx}^2\left(1-\beta_x'^2\right)\frac{\partial E'}{\partial x'}+\varepsilon_\circ\gamma_{yy}^2\left(1-\beta_y'^2\right)\frac{\partial E'}{\partial y'}+\varepsilon_\circ\gamma_{zz}^2\left(1-\beta_z'^2\right)\frac{\partial E'}{\partial z'}=\varepsilon_\circ\gamma_{x_ix_i}^2\left(1-\beta_{x_i}'^2\right)\vec{\nabla}'.\vec{E}', \tag{63}$$

where $x_i=x,y,z$. Since $\gamma_{x_ix_i}^2(1-\beta_{x_i}'^2)=1$ and $\vec{\nabla}'.\vec{E}'=\rho'/\varepsilon_\circ'$, matching both sides of Eq. (63) yields $\varepsilon_0'\rho_e=\varepsilon_0\rho_e'$. where $\varepsilon_0'=\varepsilon_0$, $\rho_e=\rho_\circ(1-\beta^2)^{-1/2}$ and $\rho_e'=\rho_\circ(1-\beta'^2)^{-1/2}$ are the charge densities in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$, defined with respect to charge density $\rho_\circ$ in frame $\bar{\Sigma}_\circ$. We can write

$$\vec{\nabla}.\vec{E}-\frac{\rho}{\varepsilon_\circ}=\vec{\nabla}'.\vec{E}'-\frac{\rho'}{\varepsilon_\circ'} \quad\rightleftarrows\quad \partial^\beta F_{0\beta}-\mu_\circ J=\partial'^\beta F_{0\beta}'-\mu_\circ' J', \tag{64}$$

which is Lorentz invariant Gauss law of electrostatics between the inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$.

7.2. Gauss Law of Magnetostatics

Since magnetic field wave function is continuous at any point in space, taking $\Phi=B$ and $\Phi'=B'$, and using chain rule in Eq. (61), we write

$$\vec{\nabla}.\vec{B}=\gamma_{xx}^2\mu_\circ\left(1-\beta_x'^2\right)\frac{\partial B'}{\partial x'}+\gamma_{yy}^2\mu_\circ\left(1-\beta_y'^2\right)\frac{\partial B'}{\partial y'}+\gamma_{zz}^2\mu_\circ\left(1-\beta_z'^2\right)\frac{\partial B'}{\partial z'}=\gamma_{x_ix_i}^2\left(1-\beta_{x_i}'^2\right)\vec{\nabla}'.\vec{B}', \tag{65}$$

Matching both sides of Eq. (74), one finds $\gamma_{x_ix_i}=1/(1-\beta_{x_i}'^2)^{1/2}$ for the space component of Lorentz scaling factor in Eq. (10), and covariant Eq. (65) is transformed into the invariant form

$$\vec{\nabla}.\vec{B}=\vec{\nabla}'.\vec{B}' \quad\rightleftarrows\quad \partial^\beta G_{0\beta}=\partial'^\beta G_{0\beta}', \tag{66}$$

which is Lorentz invariant Gauss law of magnetostatics between the inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$.

7.3. Faraday's Law of Induction

Let us re-write the differential form of Faraday's law of induction in Eq. (57b) in x, y, and z-directions of the Cartesian coordinates in frame $\bar{\Sigma}$

$$\left(\vec{\nabla}\times\vec{E}\right)_x=\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=\mu_\circ\frac{\partial H_x}{\partial t}, \quad \left(\vec{\nabla}\times\vec{E}\right)_y=\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=\mu_\circ\frac{\partial H_y}{\partial t}, \quad \left(\vec{\nabla}\times\vec{E}\right)_z=\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=\mu_\circ\frac{\partial H_z}{\partial t} \tag{67}$$

Applying the chain rule in Eqs. (61) and (62) to Faraday's law of induction allows us to write

$$\frac{\partial \vec{E}_{x_i}}{\partial x_i} = \frac{\partial \vec{E}'_{x_i}}{\partial x_i} = \frac{\partial \vec{E}'_{x_i}}{\partial x'_i}\frac{\partial x'_i}{\partial x_i} + \frac{\partial \vec{E}'_{x_i}}{\partial t'}\bigg|_{x_i}\frac{\partial t'}{\partial x_i} = \frac{\partial \vec{E}'_{x_i}}{\partial x'_i} - \beta'^2_{x_i}\frac{\partial \vec{E}'_{x_i}}{\partial x'_i}\frac{\partial x'_i}{\partial x_i}\frac{\partial x_i}{\partial t'} = \left(1-\beta'^2_{x_i}\right)\frac{\partial \vec{E}'_{x_i}}{\partial x'_i}, \tag{68a}$$

$$\frac{\partial \vec{B}_{x_i}}{\partial t} = \frac{\partial \vec{B}'_{x_i}}{\partial t} = \frac{\partial \vec{B}'_{x_i}}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial \vec{B}'_{x_i}}{\partial x'_i}\frac{\partial x'_i}{\partial t} = \frac{\partial \vec{B}'_{x_i}}{\partial t'} - \frac{\partial x'_i}{\partial t}\frac{\partial t'}{\partial t}\left(\frac{v_{x_i}}{c^2}\frac{\partial t}{\partial x'_i}\right)\frac{\partial \vec{B}'_{x_i}}{\partial t'} = \left(1-\beta'^2\right)\frac{\partial \vec{B}'_{x_i}}{\partial t'}, \tag{68b}$$

Side by side addition of Eqs. (68a) and (68b) allows us to write the covariant Faraday's law

$$\left(\vec{\nabla}\times\vec{E}\right)_{x_i} + \mu_\circ\frac{\partial \vec{H}_{x_i}}{\partial t} = \gamma^2_{x_i x_i}\left(1-\beta'^2_{x_i}\right)\left(\vec{\nabla}'\times\vec{E}'\right)_{x_i} + \gamma^2_{tt}\left(1-\beta'^2\right)\mu_\circ\frac{\partial \vec{H}'_{x_i}}{\partial t'}, \tag{69}$$

Matching both sides of Eq. (69) yield $\gamma_{x_i x_i} = 1/(1-\beta'^2_{x_i})^{1/2}$ and $\gamma_{tt} = 1/(1-\beta'^2)^{1/2}$ for components of Lorentz scaling factor in Eq. (10). Covariant Eq. (69) is then transformed into invariant form

$$\left(\vec{\nabla}\times\vec{E}\right)_{x_i} + \mu_\circ\frac{\partial \vec{H}_{x_i}}{\partial t} = \left(\vec{\nabla}'\times\vec{E}'\right)_{x_i} + \mu_\circ\frac{\partial \vec{H}'_{x_i}}{\partial t'} \quad \rightleftarrows \quad \partial^\beta G_{\alpha\beta} = \partial'^\beta G'_{\alpha\beta}, \tag{70}$$

which is Lorentz invariant in the x, y, and z-directions between the inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$.

7.4. Ampere-Maxwell Equation

Ampere-Maxwell's equation (57b) are written as

$$\left(\vec{\nabla}\times\vec{B}\right)_x = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_\circ J_x + \mu_\circ\varepsilon_\circ\frac{\partial E_x}{\partial t} = \mu_\circ\sigma E_x + \mu_\circ\varepsilon_\circ\frac{\partial E_x}{\partial t}, \tag{71a}$$

$$\left(\vec{\nabla}\times\vec{B}\right)_y = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_\circ J_y + \mu_\circ\varepsilon_\circ\frac{\partial E_y}{\partial t} = \mu_\circ\sigma E_y + \mu_\circ\varepsilon_\circ\frac{\partial E_y}{\partial t}, \tag{71b}$$

$$\left(\vec{\nabla}\times\vec{B}\right)_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_\circ J_z + \mu_\circ\varepsilon_\circ\frac{\partial E_z}{\partial t} = \mu_\circ\sigma E_z + \mu_\circ\varepsilon_\circ\frac{\partial E_z}{\partial t}, \tag{71c}$$

where $\sigma = \sigma_\circ(1-\beta^2)^{-1/2}$ and $\sigma' = \sigma_\circ(1-\beta'^2)^{-1/2}$ are the conductivities in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$, defined relative to $\sigma_\circ$ in the steady inertial frame $\Sigma_\circ$. Applying the chain rule in Eq. (61) and (62) to differential form of Ampere-Faraday's law in x, y, and z-directions and write

$$\frac{\partial \vec{B}_{x_i}}{\partial x_i} = \frac{\partial \vec{B}'_{x_i}}{\partial x_i} = \frac{\partial \vec{B}_{x_i}}{\partial x'_i}\frac{\partial x'_i}{\partial x_i} + \frac{\partial \vec{B}'_{x_i}}{\partial t'}\frac{\partial t'}{\partial x_i} = \frac{\partial \vec{B}'_{x_i}}{\partial x'_i} - \frac{v_{x_i}}{c^2}\frac{\partial \vec{B}'_{x_i}}{\partial x'_i}\frac{\partial x'_i}{\partial x_i}\frac{\partial x_i}{\partial t'} = \left(1-\beta'^2_{x_i}\right)\frac{\partial \vec{B}'_{x_i}}{\partial x'_i}, \tag{72a}$$

$$\frac{\partial \vec{E}_{x_i}}{\partial t} = \frac{\partial \vec{E}'_{x_i}}{\partial t} = \frac{\partial \vec{E}'_{x_i}}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial \vec{E}'_{x_i}}{\partial x'_i}\frac{\partial x'_i}{\partial t} = \frac{\partial \vec{E}'_{x_i}}{\partial t'} - \frac{\partial x'_i}{\partial t}\frac{\partial t'}{\partial t}\left(\frac{v_{x_i}}{c^2}\frac{\partial t}{\partial x'_i}\right)\frac{\partial \vec{E}'_{x_i}}{\partial t'} = \left(1-\beta'^2\right)\frac{\partial \vec{E}'_{x_i}}{\partial t'}, \tag{72b}$$

Combining Eqs. (72a), and (72b) side by side and adding $\mu_\circ\sigma\vec{E}_{x_i}$ and $\mu'_\circ\sigma'\vec{E}'_{x_i}$, we write

$$\left(\vec{\nabla}\times\vec{B}\right)_{x_i} - \mu_\circ\sigma\vec{E}_{x_i} + \varepsilon_\circ\frac{\partial \vec{E}_{x_i}}{\partial t} = \gamma^2_{x_i x_i}\left(1-\beta'^2_{x_i}\right)\left(\vec{\nabla}'\times\vec{B}'\right)_{x_i} - \mu'_\circ\sigma'\vec{E}'_{x_i} + \gamma^2_{tt}\varepsilon'_\circ\left(1-\beta'^2\right)\frac{\partial \vec{E}'_{x_i}}{\partial t'}, \tag{73}$$

Matching both sides of Eq. (73), one finds $\gamma_{x_i x_i} = 1/(1-\beta'^2_{x_i})^{1/2}$ and $\gamma_{tt} = 1/(1-\beta'^2)^{1/2}$ for space-time components of Lorentz scaling factor in Eq. (10). Then covariant Eq. (73) becomes invariant

$$\vec{\nabla}\times\vec{B} - \mu_\circ\sigma\vec{E} + \varepsilon_\circ\mu_\circ\frac{\partial \vec{E}}{\partial t} = \nabla'\times\vec{B}' - \mu'_\circ\sigma'\vec{E}' + \varepsilon'_\circ\mu'_\circ\frac{\partial \vec{E}'}{\partial t'} \quad \rightleftarrows \quad \partial^\beta F_{\alpha\beta} - \mu_\circ J_\alpha = \partial'^\beta F'_{\alpha\beta} - \mu'_\circ J'_\alpha, \tag{74}$$

which is Lorentz invariant Ampere-Maxwell equation between frames $\bar{\Sigma}$ and $\bar{\Sigma}'$.

7.5. Current Continuity Equation

Since Maxwell's equations must satisfy the charge (current) continuity equation, using chain rules in Eqs. (61) and (62) we can write

$$\vec{\nabla}.\vec{J} = \gamma_{xx}^2\left(\frac{\partial J'}{\partial x'} - \frac{v_x}{c'^2}\frac{\partial J'}{\partial x'}\frac{\partial x'}{\partial x}\frac{\partial x}{\partial t'}\right) + \gamma_{yy}^2\left(\frac{\partial J'}{\partial y'} - \frac{v_x}{c'^2}\frac{\partial J'}{\partial y'}\frac{\partial y'}{\partial y}\frac{\partial y}{\partial t'}\right) + \gamma_{zz}^2\left(\frac{\partial J'}{\partial z'} - \frac{v_x}{c'^2}\frac{\partial J'}{\partial z'}\frac{\partial z'}{\partial z}\frac{\partial z}{\partial t'}\right), \quad (75a)$$

$$\frac{\partial \rho}{\partial t} = \gamma_{tt}^2\frac{\partial \rho'}{\partial t'} + \gamma_{tt}^2\left(-\frac{v_x}{c^2}\frac{\partial \rho'}{\partial x'} - \frac{v_y}{c^2}\frac{\partial \rho'}{\partial y'} - \frac{v_z}{c^2}\frac{\partial \rho'}{\partial z'}\right) = \gamma_{tt}^2\left(1-\beta'^2\right)\frac{\partial \rho'}{\partial t'}, \quad (75b)$$

where $\vec{J}$ and ρ are the total current and charge densities, respectively. Side by side additions of Eqs. (75a) and (75b) allows us to write the following covariant equation

$$\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = \gamma_{x_i x_i}^2\left(1-\beta_{x_i}'^2\right)\vec{\nabla}'.\vec{J}' + \gamma_{tt}^2\left(1-\beta'^2\right)\frac{\partial \rho'}{\partial t'}, \quad (76)$$

Matching both sides of Eq. (76) we find $\gamma_{x_i x_i} = 1/(1-\beta_{x_i}'^2)^{1/2}$ and $\gamma_{tt} = 1/(1-\beta'^2)^{1/2}$ in Eq. (10), and

$$\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = \vec{\nabla}'.\vec{J}' + \frac{\partial \rho'}{\partial t'}, \qquad \rightleftarrows \quad \partial^{\alpha} J_{\alpha} = \partial'^{\alpha} J'_{\alpha} \quad (77)$$

which is the Lorentz invariant current continuity equation between frames $\bar{\Sigma}$ and $\bar{\Sigma}'$.

8. Symmetric Electromagnetic Energy-Momentum Tensor and Conservation Laws

The classical symmetric energy-momentum tensor is described by a second rank tensor [4]

$$\Theta^{\alpha\beta} = \frac{1}{\mu_\circ}\left(F^{\alpha\sigma}F_{\sigma}^{\beta} - \frac{1}{4}g^{\alpha\beta}F^{\sigma\beta}F_{\sigma\beta}\right) = \begin{pmatrix} u_{em} & S_x/c & S_x/c & S_x/c \\ S_x/c & \Theta_{xx} & \Theta_{xy} & \Theta_{xz} \\ S_x/c & \Theta_{yx} & \Theta_{yy} & \Theta_{yz} \\ S_x/c_z & \Theta_{zx} & \Theta_{zy} & \Theta_{zz} \end{pmatrix}, \quad (78)$$

where $F^{\alpha\sigma}$ is the contravariant field tensor and $F_{\beta}^{\alpha} = g^{\alpha\sigma}F_{\sigma\beta}$ $(F_{\alpha}^{\beta} = g^{\beta\sigma}F_{\alpha\sigma})$ is mixed tensor, with $g^{\mu\sigma} = g_{\mu\sigma} = diag(-1,1,1,1)$. $\Theta^{00} = u_{em}$ is energy density, $\Theta^{0\beta} = \Theta^{\beta 0} = S_{\beta}$ is Cartesian components of Poynting vector, and $\Theta_{\alpha\beta} = -T_{\alpha\beta}$, with Maxwell's stress tensor $T_{\alpha\beta}$ defined as [4]

$$T_{\alpha\beta} = \left(\varepsilon_\circ E_\alpha E_\beta + \mu_\circ^{-1}B_\alpha B_\beta\right) - \frac{1}{2}\delta_{\alpha\beta}\left(\varepsilon_\circ E^2 + \mu_\circ^{-1}B^2\right) = \left(\varepsilon_\circ E_\alpha E_\beta + \mu_\circ^{-1}B_\alpha B_\beta\right) - \delta_{\alpha\beta}u_{em}, \quad (79)$$

where $\delta_{\alpha\beta}$ is the Kronecker delta which is unity if $\delta_{xx} = \delta_{yy} = \delta_{zz}$ and zero otherwise [4]. Mixed tensor $F_{\beta}^{\alpha} = g^{\alpha\sigma}F_{\sigma\beta}$ $(F_{\alpha}^{\beta} = g^{\beta\sigma}F_{\alpha\sigma})$ in Eq. (78) has no explicit symmetry characteristics [4].

Consequently, we pursued a search for an alternative way to derive Lorentz invariant expression for a symmetric electromagnetic energy-momentum tensor. We discovered that the average sum of the product of covariant-contravariant Faraday tensor and transpose of the contravariant covariant dual Faraday tensors $\bar{T}^{\alpha\beta} = (F_{\alpha\beta}\tilde{F}^{\alpha\beta} + G_{\alpha\beta}\tilde{G}^{\alpha\beta})/2$ fits our requirement of the electromagnetic energy-momentum tensor. In the massive inertial frame $\bar{\Sigma}$ they are written as

$$F_{\alpha\beta}\tilde{F}^{\alpha\beta}=\begin{pmatrix} -\frac{i^2}{c^2}\vec{E}^2 & \frac{i}{c}\mu_\circ S_x & \frac{i}{c}\mu_\circ S_y & \frac{i}{c}\mu_\circ S_z \\ \frac{i}{c}\mu_\circ S_x & \left(-\frac{i^2}{c^2}E_x^2-B_y^2-B_z^2\right) & \left(-\frac{i^2}{c^2}E_xE_y+B_xB_y\right) & \left(-\frac{i^2}{c^2}E_xE_z+B_xB_z\right) \\ \frac{i}{c}\mu_\circ S_y & \left(-\frac{i^2}{c^2}E_yE_x+B_yB_x\right) & \left(-\frac{i^2}{c^2}E_y^2-B_z^2-B_x^2\right) & \left(-\frac{i^2}{c^2}E_yE_z+B_yB_z\right) \\ \frac{i}{c}\mu_\circ S_z & \left(-\frac{i^2}{c^2}E_zE_x+B_zB_x\right) & \left(-\frac{i^2}{c^2}E_zE_y+B_zB_y\right) & \left(-\frac{i^2}{c^2}E_z^2-B_x^2-B_y^2\right) \end{pmatrix}, \tag{80}$$

$$G_{\alpha\beta}\tilde{G}^{\alpha\beta}=\begin{pmatrix} -i^2\vec{B}^2 & \frac{i}{c}\mu_\circ S_x & \frac{i}{c}\mu_\circ S_y & \frac{i}{c}\mu_\circ S_z \\ \frac{i}{c}\mu_\circ S_x & \left(-i^2B_x^2-\frac{1}{c^2}E_y^2-\frac{1}{c^2}E_z^2\right) & \left(\frac{1}{c^2}E_xE_y-i^2B_xB_y\right) & \left(\frac{1}{c^2}E_xE_z-i^2B_xB_z\right) \\ \frac{i}{c}\mu_\circ S_y & \left(\frac{1}{c^2}E_yE_x-i^2B_yB_x\right) & \left(-i^2B_y^2-\frac{1}{c^2}E_z^2-\frac{1}{c^2}E_x^2\right) & \left(\frac{1}{c^2}E_zE_y-i^2B_yB_x\right) \\ \frac{i}{c}\mu_\circ S_z & \left(\frac{1}{c^2}E_zE_x-i^2B_zB_x\right) & \left(\frac{1}{c^2}E_zE_y-i^2B_zB_y\right) & \left(-i^2B_z^2-\frac{1}{c^2}E_x^2-\frac{1}{c^2}E_y^2\right) \end{pmatrix}, \tag{81}$$

which have symmetric characteristics and consequently, algebraic sum of their traces is zero

$$Tr\left(F_{\alpha\beta}\tilde{F}^{\alpha\beta}\right)+Tr(G_{\alpha\beta}\tilde{G}^{\alpha\beta})=\left(2\frac{E^2}{c^2}-2B^2\right)-\left(2\frac{E^2}{c^2}-2B^2\right)=0 \tag{82}$$

Average sum of the product of covariant (contravariant) and transpose of contravariant covariant) Faraday tensors and of their duals, $\bar{T}^{\alpha\beta}=(F_{\alpha\beta}\tilde{F}^{\alpha\beta}+G_{\alpha\beta}\tilde{G}^{\alpha\beta})/2$ in frame $\bar{\Sigma}$ is then written as

$$\bar{T}^{\alpha\beta}=$$
$$\begin{pmatrix} \mu_\circ u_{em} & \frac{i}{c}\mu_\circ S_x & \frac{i}{c}\mu_\circ S_y & \frac{i}{c}\mu_\circ S_z \\ \frac{i}{c}\mu_\circ S_x & B_x^2-\frac{i^2}{c^2}E_x^2-\frac{1}{2}\left(\vec{B}^2-\frac{i^2}{c^2}\vec{E}^2\right) & -\frac{i^2}{c^2}E_xE_y+B_xB_y & -\frac{i^2}{c^2}E_xE_z+B_xB_z \\ \frac{i}{c}\mu_\circ S_y & -\frac{i^2}{c^2}E_xE_y+B_xB_y & B_y^2-\frac{i^2}{c^2}E_y^2-\frac{1}{2}\left(\vec{B}^2-\frac{i^2}{c^2}\vec{E}^2\right) & -\frac{i^2}{c^2}E_yE_z+B_yB_z \\ \frac{i}{c}\mu_\circ S_z & -\frac{i^2}{c^2}E_xE_z+B_xB_z & -\frac{i^2}{c^2}E_yE_z+B_yB_z & B_z^2-\frac{i^2}{c^2}E_z^2-\frac{1}{2}\left(\vec{B}^2-\frac{i^2}{c^2}\vec{E}^2\right) \end{pmatrix} \tag{83}$$
$$=\bar{T}^{\beta\alpha},$$

Cartesian components of the symmetric energy-momentum tensor $\bar{T}^{\alpha\beta}$ in Eq. (82) are

$$\bar{T}^{00}=\mu_\circ u_{em}=\frac{1}{2}\mu_\circ\left(\varepsilon_\circ\vec{E}^2+\mu_\circ^{-1}\vec{B}^2\right) \quad \text{(Electromagnetic energy density),} \tag{84a}$$

$$\bar{T}^{0\beta}=\frac{i}{c}\mu_\circ\vec{S}_\beta=\frac{i}{c}\left(\vec{E}\times\vec{B}\right)_\beta \quad \text{(x, y, and z-components of Poynting vector)} \tag{84b}$$

$$T^{\beta 0}=(i/c)\mu_\circ\vec{S}_\beta=(i/c)\mu_\circ c^2\vec{g}_\beta \quad \text{(x, y, and z-components of momentum density)} \tag{84c}$$

where $T_{\alpha\beta}$ is given by Eq. (79). Since $\bar{T}^{\alpha\beta}$ is symmetric $(\bar{T}^{\alpha\beta}=\bar{T}^{\beta\alpha})$ its trace must be zero:

$$Tr(\bar{T}^{\alpha\beta})=\bar{T}^{00}+\bar{T}^{11}+\bar{T}^{22}+\bar{T}^{33}=\mu_\circ u_{em}+\left(\vec{B}^2-\frac{3}{2}\vec{B}^2\right)-\frac{i^2}{c^2}\left(\vec{E}^2-\frac{3}{2}\vec{E}^2\right)=\mu_\circ u_{em}-\mu_\circ u_{em}=0 \tag{85}$$

9. Poynting Theorem and Conservation Laws

Using $\vec{\nabla}\cdot(\vec{a}\times\vec{b})=\vec{b}\cdot(\vec{\nabla}\times\vec{a})-\vec{a}\cdot(\vec{\nabla}\times\vec{b})$ and Maxwell's equations (56) and (57), divergence of Poynting vector can be written as

$$\begin{aligned}\vec{\nabla}\cdot\vec{S}&=\frac{1}{\mu_\circ}\vec{\nabla}\cdot(\vec{E}\times\vec{B})=-\frac{1}{\mu_\circ}\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)+\frac{1}{\mu_\circ}\vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right)=-\vec{E}\cdot\left[\vec{J}+\varepsilon_\circ\frac{\partial\vec{E}}{\partial t}\right]-\vec{H}\cdot\left[\mu_\circ\frac{\partial\vec{H}}{\partial t}\right]\\&=-\vec{E}\cdot\vec{J}_e-\frac{\partial}{\partial t}\left(\frac{\varepsilon_\circ}{2}\vec{E}\cdot\vec{E}+\frac{1}{2\mu_\circ}\vec{B}\cdot\vec{B}\right)=-\vec{E}\cdot\vec{J}_e-\frac{\partial u_{em}}{\partial t},\end{aligned}\tag{86}$$

where, $\vec{E}\cdot\vec{J}=\partial W/\partial t$ is the rate of work done by Lorentz force on moving charged particles and $u_{em}=(\varepsilon_\circ\vec{E}\cdot\vec{E}+\mu_\circ^{-1}\vec{B}\cdot\vec{B})/2$ is the electromagnetic energy density. Divergence theorem yields

$$\oint_S\left(\vec{E}\times\vec{B}\right)\cdot d\vec{a}+\int_v\frac{\partial u_{em}}{\partial t}dv=-\int_v\left(\vec{E}.\vec{J}\right)dv,\tag{87}$$

which is the well-known classical Poynting theorem [17]. The first and second terms on the left are the power flowing out of the volume and rate of stored energy density and the term on right is the power dissipated (or generated) in a closed electrical circuit.

We now demonstrate the use of symmetric energy-momentum tensor $\bar{T}^{\alpha\beta}$ allows us to derive differential form of Poynting theorem and conservation of linear momentum with charge and current sources. Using the 4-vector current density $J_\alpha=(-ic\rho,\vec{J})$ in $\partial_\beta\bar{T}^{\alpha\beta}=\mu_\circ F^{\alpha\beta}J_\beta$ we write

$$\begin{pmatrix}(i/c)\mu_\circ\dfrac{\partial u_{em}}{\partial(ict)} & (i/c)\mu_\circ\dfrac{\partial S_x}{\partial x} & (i/c)\mu_\circ\dfrac{\partial S_y}{\partial y} & (i/c)\mu_\circ\dfrac{\partial S_z}{\partial z}\\ (i/c)c^2\mu_\circ\dfrac{\partial g_x}{\partial(ict)} & \mu_\circ\dfrac{\partial T_{xx}}{\partial x} & \mu_\circ\dfrac{\partial T_{xy}}{\partial y} & \mu_\circ\dfrac{\partial T_{xz}}{\partial z}\\ (i/c)c^2\mu_\circ\dfrac{\partial g_y}{\partial(ict)} & \mu_\circ\dfrac{\partial T_{yx}}{\partial x} & \mu_\circ\dfrac{\partial T_{yy}}{\partial y} & \mu_\circ\dfrac{\partial T_{yz}}{\partial z}\\ (i/c)c^2\mu_\circ\dfrac{\partial g_z}{\partial(ict)} & \mu_\circ\dfrac{\partial T_{zx}}{\partial x} & \mu_\circ\dfrac{\partial T_{zy}}{\partial y} & \mu_\circ\dfrac{\partial T_{zz}}{\partial z}\end{pmatrix}=\begin{pmatrix}-(i/c)\mu_\circ\left(E_xJ_x+E_yJ_y+E_zJ_z\right)\\ -\mu_\circ\rho E_x+\mu_\circ\left(B_yJ_z-B_zJ_y\right)\\ -\mu_\circ\rho E_y+\mu_\circ\left(B_zJ_x-B_xJ_z\right)\\ -\mu_\circ\rho E_x+\mu_\circ\left(B_xJ_y-B_yJ_x\right)\end{pmatrix},\tag{88}$$

Sum of the first-row elements on both sides of Eq. (88) yields the energy conservation law

$$\mu_\circ\left(\frac{\partial u_{em}}{\partial t}+\vec{\nabla}\cdot\vec{S}\right)=-\mu_\circ\left(E_xJ_x+E_yJ_y+E_zJ_z\right)\qquad\Rightarrow\qquad\frac{\partial u_{em}}{\partial t}+\vec{\nabla}\cdot\vec{S}=-\vec{E}\cdot\vec{J},\tag{89}$$

which is the differential form of the Poynting theorem in free space given by Eq. (87). Further, adding the second, third and fourth row elements of Eq. (88) and combining them we write

$$\frac{i}{c}\left(\mu_\circ c^2\right)\frac{\partial g_\beta}{\partial(ict)}+\mu_0\sum_{j\neq0}^{3}\frac{\partial T_{ij}}{\partial x_j}=-\mu_0\sum_{\beta\neq0}^{3}\left(F^{\alpha\beta}J_\beta^e+G^{\alpha\beta}J_\beta^m\right)\quad\Rightarrow\quad\frac{\partial\vec{g}}{\partial t}+\vec{\nabla}\cdot\vec{T}=-\left(\rho\vec{E}+\vec{J}\times\vec{B}\right),\tag{90}$$

which is the differential form of conservation of linear momentum with total current source.

Since for planar rotation about a fixed axis, Poynting vector is Lorentz invariant $(\vec{S}'=\vec{S})$, letting $\Phi=S$ (or u_{em}) and $\Phi'=S'$ (or u'_{em}), Eq. (89) can be decomposed as

$$\vec{\nabla}.\vec{S}=\gamma_{xx}^2\left(1-\beta_x'^2\right)\frac{\partial S'}{\partial x'}+\gamma_{yy}^2\left(1-\beta_y'^2\right)\frac{\partial S'}{\partial y'}+\gamma_{zz}^2\left(1-\beta_y'^2\right)\frac{\partial S'}{\partial z'},\tag{91a}$$

$$\frac{\partial u_{em}}{\partial t}=\gamma_{tt}^2\frac{\partial u'_{em}}{\partial t'}-\gamma_{tt}^2\left(\frac{v_x}{c^2}\frac{\partial t}{\partial x'}+\frac{v_y}{c^2}\frac{\partial t}{\partial y'}+\frac{v_z}{c^2}\frac{\partial t}{\partial z'}\right)\frac{\partial u'_{em}}{\partial t'}=\gamma_{tt}^2\left(1-\beta'^2\right)\frac{\partial u'_{em}}{\partial t'},\tag{91b}$$

$$\vec{E}\cdot\vec{J}=\vec{E}'\cdot\vec{J}',\tag{91c}$$

Side by side adding Eqs. (91a), (91b), and (91c) yields the following covariant energy continuity equation

$$\vec{\nabla}.\vec{S}+\frac{\partial u_{em}}{\partial t}+\vec{E}\cdot\vec{J}=\gamma_{x_ix_i}^2\left(1-\beta_{x_i}'^2\right)\vec{\nabla}'.\vec{S}'+\gamma_{tt}^2\left(1-\beta'^2\right)\frac{\partial u_{em}'}{\partial t'}+\vec{E}'\cdot\vec{J}', \tag{92}$$

which yield $\gamma_{x_ix_i}=1/(1-\beta_{x_i}'^2)^{1/2}$ and $\gamma_{tt}=1/(1-\beta'^2)^{1/2})^{1/2}$ in Eq. (10), which transforms covariant equation Eq. (92) into the following invariant form

$$\vec{\nabla}.\vec{S}+\frac{\partial u_{em}}{\partial t}=\vec{\nabla}.\vec{S}'+\frac{\partial u_{em}'}{\partial t'}, \tag{93}$$

Similarly, employing the differentiation chain rule to differential form of conservation of linear momentum equation with current source in Eq. (90) allows one to write the following equations

$$\frac{\partial\vec{g}}{\partial t}=\frac{\partial\vec{g}'}{\partial t}=\frac{\partial\vec{g}'}{\partial t'}\frac{\partial t'}{\partial t}+\frac{\partial\vec{g}}{\partial x_i'}\frac{\partial x_i'}{\partial t}=\frac{\partial\vec{g}'}{\partial t'}-\frac{\partial x_i'}{\partial t}\frac{\partial t'}{\partial t}\left(\frac{v_{x_i}}{c^2}\frac{\partial t}{\partial x_i'}\right)\frac{\partial\vec{g}'}{\partial t'}=\gamma_{tt}^2\left(1-\beta^2\right)\frac{\partial\vec{g}'}{\partial t'}, \tag{94a}$$

$$\vec{\nabla}.\vec{T}=\gamma_{xx}^2\left(1-\frac{v_x^2}{c'^2}\right)\frac{\partial T_{\alpha\beta}'}{\partial x'}+\gamma_{yy}^2\left(1-\frac{v_y^2}{c'^2}\right)\frac{\partial T_{\alpha\beta}'}{\partial y'}+\gamma_{zz}^2\left(1-\frac{v_z^2}{c'^2}\right)\frac{\partial T_{\alpha\beta}'}{\partial z'}, \tag{94b}$$

$$\vec{f}=\rho\vec{E}+\vec{J}\times\vec{B}=\rho'\vec{E}'+\sigma'\left(\vec{E}'\times\vec{B}'\right)=\vec{f}', \tag{94c}$$

Adding Eqs. (94a), (94b), and (94c) yields the covariant momentum continuity equation

$$\frac{\partial\vec{g}}{\partial t}+\vec{\nabla}.\vec{T}+\vec{f}=\gamma_{tt}^2\left(1-\beta^2\right)\frac{\partial\vec{g}'}{\partial t'}+\gamma_{x_ix_i}^2\left(1-\frac{v_{x_i}^2}{c'^2}\right)\frac{\partial T_{\alpha\beta}'}{\partial x_i'}+\vec{f}', \tag{95}$$

Matching both sides of Eq. (95), one finds $\gamma_{x_ix_i}=1/(1-v_{x_i}^2/c'^2)^{1/2}$ and $\gamma_{tt}=1/(1-v^2/c'^2)^{1/2}$ in Eq. (10) for components of Lorentz factor and covariant Eq. (95) is transformed invariant form

$$\frac{\partial\vec{g}}{\partial t}+\vec{\nabla}\cdot\vec{T}+\left(\rho\vec{E}+\vec{J}\times\vec{B}\right)=\frac{\partial\vec{g}'}{\partial t'}+\vec{\nabla}'\cdot\vec{T}'+\left(\rho'\vec{E}'+\vec{J}\times\vec{B}'\right), \tag{96}$$

which is Lorentz invariant linear momentum conservation equation between frames $\bar{\Sigma}$ and $\bar{\Sigma}'$.

10. Results and Discussions

In this work, we introduced so called generalized 4-dimensional massive inertial frames $\bar{\Sigma}'=\bar{\Sigma}'\left(x',y',z',ic't'\right)$ and $\bar{\Sigma}=\bar{\Sigma}\left(x,y,z,ict\right)$, both coincide with stationary inertial frame $\Sigma_0=\Sigma_0\left(x_0,y_0,z_0,t_0\right)$ at time $t'=t=t_\circ=0$, and move relative to each other with arbitrary velocity $\vec{v}=(v_x,v_y,v_z)$. The space and time coordinates are interrelated: $x_i'=x_i'(x_i,t)$, $t'=t'(t,x,y,z)$ in $\bar{\Sigma}'$ and $x_i=x_i(x_i',t')$, $t=t(t',x',y',z')$ in frame $\bar{\Sigma}$, where $i=x,y,z$. We derived the generalized 4-dimensional spacetime metric equation (8), with Lorentz scaling factor in Eq. (10), which has anisotropic space and uniform time components. The role of this factor on the invariance of relativistic quantities such as energy, momentum, mass, time dilation and Doppler shift is discussed in appendix. In the following, we discuss Lorentz invariance of electromagnetic fields, Maxwell's equations, and symmetric electromagnetic field energy-momentum tensor.

10.1. Comparison of Classical and New Way of Using Faraday Tensor in Field Transformation

By using the 4- velocity and 4-force vectors in the energy conservation law we demonstrated the Lorentz invariance of the scalar and vector products of electric and magnetic fields between two so called massive inertial frames for a spatial rotation about a fixed axis. As an example, consider point charge Q in frame $\bar{\Sigma}$. The produced electric and magnetic fields in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ are

$$\vec{E}=\frac{q}{4\pi\varepsilon_\circ}\frac{\hat{r}}{r^2}=\frac{Q}{4\pi\varepsilon_\circ r^{3/2}}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=E_x\hat{i}+E_y\hat{j}+E_z\hat{k}, \tag{97a}$$

$$\vec{E}' = \frac{q}{4\pi\varepsilon_\circ}\frac{\hat{r}'}{r'^2} = \frac{Q}{4\pi\varepsilon_\circ r'^{3/2}}\left[x'\hat{i}' + y'\hat{j}' + z'\hat{k}'\right] = \frac{Q}{4\pi\varepsilon_\circ r^{3/2}}\left[x\hat{i} + y\hat{j} + z\hat{k}\right] = \vec{E}, \tag{97b}$$

$$\vec{B} = \frac{\mu_\circ}{4\pi}\frac{q\left(\vec{u}\times\hat{r}\right)}{r^2} = \frac{\mu_\circ}{4\pi}\frac{q}{r^{3/2}}\left(u_x\hat{i} + u_y\hat{j} + u_z\hat{k}\right)\times\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}, \tag{97c}$$

$$\begin{aligned}\vec{B}' &= \frac{\mu_\circ}{4\pi}\frac{q\left(\vec{u}'\times\hat{r}'\right)}{r'^2} = \frac{\mu_\circ}{4\pi}\frac{q}{r^{3/2}}\left[\left(u'_y z' - u'_z y'\right)\hat{i}' + \left(-u'_x z' + u'_z x'\right)\hat{j}' + \left(u'_x y' - u'_y x'\right)\hat{k}'\right] \\ &= \left(B'_x\cos\phi - B'_y\sin\phi\right)\hat{i} + \left(B'_x\sin\phi + B'_y\cos\phi\right)\hat{j} + B'_z\hat{k} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} = \vec{B},\end{aligned} \tag{97d}$$

where $r'^2 = r^2$. One can show that the scalar and vector products of electric and magnetic fields are Lorentz invariant $(\vec{E}'\cdot\vec{B}' = \vec{E}\cdot\vec{B})$ and $(\vec{E}'\times\vec{B}' = \vec{E}\times\vec{B})$ between two massive inertial frames.

To confirm the findings in section 4, we focused on the use of electromagnetic field strength tensor, also called Faraday tensor in sections 5 and 6 to derive expressions for the invariant electromagnetic fields. We demonstrated that the hyperbolic boost along x-axis classical use of Faraday tensor in field transformation $F'_{\mu\nu,z} = \Lambda_z F_{\mu\nu}\tilde{\Lambda}_z$ given Eq. (6) and $F'_{\alpha\beta,z} = L_z F_{\alpha\beta}\tilde{L}_z$ (45) for Lorentz boosts along the x, y and z-axes, lead to a non-invariant vector product of electric and magnetic fields, and in turn non-invariant Poynting vector, between two inertial frames. For a Lorentz boost along x-axis, one can use the transformation $F'_{\alpha\beta,z} = L_z F_{\alpha\beta}\tilde{L}_z$ to find components of Faraday tensor in frame $\bar{\Sigma}'$ in terms of those in frame $\bar{\Sigma}$. Transformation $F'_{\alpha\beta,z} = L_z F_{\alpha\beta}\tilde{L}_z$ and its inverse $F_{\alpha\beta,z} = \tilde{L}_z F'_{\alpha\beta} L_z$ yield Cartesian components of the electric and magnetic fields in frames $\bar{\Sigma}'$ and $\bar{\Sigma}$ in terms of each other, given by Eqs. (47a), (47b), (47c), and (47d), respectively. They are the same as Eqs. (4a) and (4b), with $\vec{E}'\cdot\vec{B}' = \vec{E}\cdot\vec{B}$ and $\vec{E}'\times\vec{B}' \neq \vec{E}\times\vec{B}$. In zero velocity case, Eqs. (47a), (47b), (47c), and (47d) results in invariant vector product. However, this has no relativistic meaning since zero velocity case means that two frames coincide with each other at rest.

In the case of circular Lorentz boosts about the x, y, and z-axes given by Eq. (48), the result of Faraday tensor transformations surprisingly turns out to be quite different. For example, the circular Lorentz boost along x-axis with fixed z-axis, transformation $F'_{\alpha\beta,z}(\theta) = L_z(\theta)F_{\alpha\beta}\tilde{L}_z(\theta)$ yields components of covariant Faraday tensor $F'_{\alpha\beta,z}(\theta)$ in frame $\bar{\Sigma}'$, which leads to Cartesian components of electromagnetic fields in frame $\bar{\Sigma}'$ as mixture of those in frame $\bar{\Sigma}$ at any angle, according to Eqs. (50). However, as easily seen from Eqs. (51) and (52) for $\theta = 0^\circ$, contrary to the hyperbolic Lorentz boost in Eq. (47), the vector products of electric and magnetic fields is Lorentz invariant ($\vec{E}'\times\vec{B}' = \vec{E}\times\vec{B}$) between frames $\bar{\Sigma}$ and $\bar{\Sigma}'$. This may be due to the difference between hyperbolic boosts in Eq. (45) and circular boost Eq. (48), which are derived from translational and rotational points of views for which $x^2 - y^2 = 1$ and $x^2 + y^2 = 1$, respectively, in two dimensions.

In section 6.2 we demonstrated that if two frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ are related by spatial rotation of (x, y) plane with fixed z-axis, of *(y, z)* plane about x axis, and *(z, x)* plane about y axis, respectively, Cartesian components of covariant (contravariant) Faraday tensor $F'_{\alpha\beta}(F'^{\alpha\beta})$ in frame $\bar{\Sigma}'$ can be found according to the spatial rotation transformations given by Eq. (52). For a spatial rotation of (x,y) plane about the z-axis, covariant Faraday tensor in frame $\bar{\Sigma}'$ is obtained from $F'_{\alpha\beta,z}(\theta) = R_z(\theta)F_{\alpha\beta}\tilde{R}_z(\theta)$, which yields Lorentz invariant electric and magnetic fields in frame $\bar{\Sigma}'(\bar{\Sigma})$ in terms of those in frame $\bar{\Sigma}(\bar{\Sigma}')$ according to Eqs. (54a) - (54d). Similar results are found under the spatial rotation of *(y, z)* plane about x-axis, and *(z, x)* plane about the y-axis, respectively.

10.2. Faraday Tensor and Lagrange Density of Electromagnetic Field

In section 6 we also proved that the new way of using Faraday transformation under the spatial rotations about fixed coordinate axes lead to Lorentz invariant properties (e.g., Poynting vector, inhomogeneous and homogeneous Maxwell's equations, and equations of continuity) between two frames $\bar{\Sigma}$ and $\bar{\Sigma}'$, respectively. Considering a charged particles moving under the influence of an external electromagnetic field we use so called the action principle $S = \int L dx^4 = \int L dx^3 dt$ to derive its equation of motion [18]. Here *L* is the total Lagrange density of the charged particle which is equal to sum of the free space (L_{em}) and external source (L_{ext}) contributions and is written as

$$L = L_{em} + L_{ext} = \frac{-1}{4\mu_\circ}F_{\alpha\beta}F^{\alpha\beta} + J^\alpha A_\alpha, \qquad (\alpha,\beta = 0,1,2,3) \tag{98}$$

According to Eqs. (43a) and (43b), the trace of the product of $F_{\alpha\beta}F^{\alpha\beta}$ and of their duals $G_{\alpha\beta}G^{\alpha\beta}$ are non-zero and yield a nonzero electromagnetic energy density in free space. Consequently, we write

$$L_{em} = \frac{-1}{4\mu_\circ}F_{\alpha\beta}F^{\alpha\beta} = \frac{-1}{4\mu_\circ}G_{\alpha\beta}G^{\alpha\beta} \quad \Rightarrow \quad Tr\left(\frac{-1}{4\mu_\circ}F_{\alpha\beta}F^{\alpha\beta}\right) = Tr\left(\frac{-1}{4\mu_\circ}G_{\alpha\beta}G^{\alpha\beta}\right) = u_{em}, \tag{99}$$

which proves that contrary to the classical point of view [4], the electromagnetic Lagrange density L_{em} is not zero in free space. This is physically realistic because electromagnetic waves transfer energy and momentum, which are the intrinsic properties of free space. Trace of product of covariant and contravariant Faraday tensors and of their duals in frames $\bar{\Sigma}$ and $\bar{\Sigma}'$ are

$$-Tr\left(F_{\alpha\beta}F^{\alpha\beta}\right) = Tr\left(G_{\alpha\beta}G^{\alpha\beta}\right) = -2\left(\frac{i^2}{c^2}\vec{E}^2 - \vec{B}^2\right) = 4\mu_\circ\left(\frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_\circ}\vec{B}^2\right) = 4\mu_\circ u_{em}, \tag{100a}$$

$$-Tr\left(F'^{\alpha\beta}F'_{\alpha\beta}\right) = Tr\left(G'^{\alpha\beta}G'_{\alpha\beta}\right) = -2\left(\frac{i^2}{c'^2}\vec{E}'^2 - \vec{B}'^2\right) = 4\mu'_\circ\left(\frac{\varepsilon'_0}{2}\vec{E}'^2 + \frac{1}{2\mu'_\circ}\vec{B}'^2\right) = 4\mu'_\circ u'_{em}, \tag{100b}$$

which show that the product of $F_{\alpha\beta}F^{\alpha\beta}$ and of their duals $G_{\alpha\beta}G^{\alpha\beta}$ are Lorentz invariant between frames $\bar{\Sigma}$ and $\bar{\Sigma}'$. Meanwhile the trace of product of covariant (contravariant) Faraday tensors and of dual Faraday tensors are also invariant between the massive inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$, written as

$$Tr\left(F_{\alpha\beta}G^{\alpha\beta}\right) = Tr\left(F^{\alpha\beta}G_{\alpha\beta}\right) = \frac{4}{c}\left(\vec{E}\cdot\vec{B}\right), \quad Tr\left(F'_{\alpha\beta}G'^{\alpha\beta}\right) = Tr\left(F'^{\alpha\beta}G'_{\alpha\beta}\right) = \frac{4}{c'}\left(\vec{E}'\cdot\vec{B}'\right), \tag{101}$$

which is zero when $\vec{E}\cdot\vec{B} = \vec{E}'\cdot\vec{B}' = 0$, and $F_{\alpha\beta}G_{\alpha\beta} = F'_{\alpha\beta}G'_{\alpha\beta}$, $F^{\alpha\beta}G^{\alpha\beta} = F'^{\alpha\beta}G'^{\alpha\beta}$ are invariant.

10.3. Symmetric Electromagnetic Energy-Momentum Tensor and Conservation Laws

In section 9 we used the symmetric electromagnetic field energy-momentum tensor to prove the invariance of the conservation of electromagnetic energy and linear momentum. We now extend this idea to cases under spatial rotations of planes about a fixed axis. For example, applying the 4-vector differentiation operator $\partial_\alpha = \partial/\partial x^\alpha = (\partial/\partial(ict), \vec{\nabla})$, we can extend Eq. (93) to

$$\begin{pmatrix} -i\partial u_{em}/c\partial t \\ \partial S_x/\partial x \\ \partial S_y/\partial y \\ \partial S_z/\partial z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -i\partial u'_{em}/c\partial t \\ \partial S'_x/\partial x \\ \partial S'_y/\partial y \\ \partial S'_z/\partial z \end{pmatrix} = R_z(-\theta)\begin{pmatrix} -i\partial u'_{em}/c\partial t \\ \partial S'_x/\partial x \\ \partial S'_y/\partial y \\ \partial S'_z/\partial z \end{pmatrix} \tag{102}$$

and using the chain rule in Eqs. (61) and (62 for the space and time differential operators, Eq. (102) can be written as covariant differential form of Poynting theorem with no current source

$$\frac{\partial u_{em}}{\partial t} + \vec{\nabla}.\vec{S} = \gamma_{tt}^2\left(1-\beta'^2\right)R_z(-\theta)\frac{\partial u'_{em}}{\partial t'} + \gamma_{x_ix_i}^2\left(1-\beta'^2_{x_i}\right)R_z(-\theta)\vec{\nabla}.\vec{S}', \tag{103}$$

Matching both sides of Eq. (103) one finds $\gamma_{x_ix_i} = R_z(\theta)/(1-\beta_{x_i}^2)^{1/2}$ and $\gamma_{tt} = R_z(\theta)/(1-\beta'^2)^{1/2}$ for the spatial rotation dependent space and time components of Lorentz scaling factor, which reduce to those in Eq. (10) without rotation, and transforms covariant Eq. (103) into invariant form

$$\vec{\nabla}.\vec{S} + \frac{\partial u_{em}}{\partial t} = \vec{\nabla}.\vec{S}' + \frac{\partial u'_{em}}{\partial t'}, \tag{104}$$

which is the Lorentz invariant classical differential form of Poynting theorem in free space.

Using the 4-vector operator $\partial_\alpha = \partial/\partial x^\alpha = (\partial/\partial(ict), \vec{\nabla})$ and chain rule in Eqs. (61) and (62) we can extend Eq. (95) to the following equation for conservation of linear momentum

$$\begin{pmatrix} -i\partial g_x / \partial(ct) \\ \partial T_{xx} / \partial x \\ \partial T_{xy} / \partial x \\ \partial T_{xz} / \partial x \end{pmatrix} = R_z(-\theta) \begin{pmatrix} -i\partial g'_x / \partial(ct) \\ \partial T'_{xx} / \partial x \\ \partial T'_{xy} / \partial y \\ \partial T'_{xz} / \partial z \end{pmatrix} = R_z(-\theta) \left(\gamma_{tt}^2 \left(1-\beta^2\right) \frac{\partial \vec{g}'_x}{\partial t'} + \gamma_{x_i x_i}^2 \left(1 - \frac{v_{x_i}^2}{c'^2}\right) \sum_{\beta}^{x,y,z} \frac{\partial T'_{x\beta}}{\partial x'_i} \right), \tag{105}$$

Matching both sides of Eq. (105) one finds $\gamma_{x_i x_i} = R_z(\theta)/(1-\beta_{x_i}^2)^{1/2}$ and $\gamma_{tt} = R_z(\theta)/(1-\beta'^2)^{1/2}$ for space and time components of Lorentz scaling factor, which reduce to those in Eq. (10) without spatial rotation and transforms covariant Eq. (105) into invariant form. Following the similar steps to produce Eq. (104), we can write the conservation equations along the y- and z-components of linear momentum. Adding x, y and z components we write the following differential equation for 4-vector linear momentum conservation in the generalized 4-dimensional spacetime frame

$$\frac{\partial \vec{g}}{\partial t} + \vec{\nabla}.\vec{T} = \frac{\partial \vec{g}'}{\partial t'} + \vec{\nabla}' \cdot \vec{T}', \tag{106}$$

which is the Lorentz invariant differential form of linear momentum conservation in free space.

10.4. Symmetric Electromagnetic Energy-Momentum Tensor for Field Transformation

It is also instructive to see if the use of the symmetric electromagnetic field energy-momentum tensor can be used to transform electromagnetic fields between two massive inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$. We try this by using the counterclockwise rotation of (x, y) plane about the z-axis in Eq. (52) and the transformation $\bar{T}'^{\alpha\beta} = R_z(\theta)\bar{T}^{\alpha\beta}\tilde{R}_z(\theta)$ yields the following symmetric tensor $\bar{T}'^{\alpha\beta}$ in frame $\bar{\Sigma}'$

$$\bar{T}'^{\alpha\beta} = \begin{pmatrix} \mu_\circ u_{em} & \frac{i}{c}\mu_\circ\left(S_x\cos\theta + S_y\sin\theta\right) & \frac{i}{c}\mu_\circ\left(S_y\cos\theta - S_x\sin\theta\right) & \frac{i}{c}\mu_\circ S_z \\ \frac{i}{c}\mu_\circ\left(S_x\cos\theta + S_y\sin\theta\right) & -\mu_\circ T'_{xx} & -\mu_\circ T'_{xy} & -\mu_\circ T'_{xz} \\ \frac{i}{c}\mu_\circ\left(S_y\cos\theta - S_x\sin\theta\right) & -\mu_\circ T'_{yx} & -\mu_\circ T'_{yy} & -\mu_\circ T'_{yz} \\ \frac{i}{c}\mu_\circ S_z & -\mu_\circ T'_{zx} & -\mu_\circ T'_{zy} & -\mu_\circ T'_{zz} \end{pmatrix}, \tag{107}$$

$$u'_{em} = u_{em}, \quad S'_x = S_x\cos\theta + S_y\sin\theta, \quad S'_y = S_y\cos\theta - S_x\sin\theta, \quad S'_z = S_z, \tag{108a}$$

$$g'_x = \frac{1}{c^2}\left(S_x\cos\theta + S_y\sin\theta\right), \quad g'_y = \frac{1}{c^2}\left(S_y\cos\theta - S_x\sin\theta\right), \quad g'_z = \frac{1}{c^2}S_z, \tag{108b}$$

$$\begin{aligned} &T'_{xx} = T_{xx}\cos^2\theta + T_{yy}\sin^2\theta + 2T_{xy}\sin\theta\cos\theta, \quad T'_{xz} = T'_{zx} = T_{xz}\cos\theta + T_{yz}\sin\theta, \\ &T'_{yy} = T_{yy}\cos^2\theta + T_{xx}\sin^2\theta - 2T_{xy}\sin\theta\cos\theta, \quad T'_{yz} = T'_{zy} = -T_{xz}\sin\theta + T_{yz}\cos\theta, \\ &T'_{xy} = T_{xy}\cos^2\theta - T_{yx}\sin^2\theta - (T_{xx} - T_{yy})\sin\theta\cos\theta, \qquad T'_{zz} = T_{zz} \\ &T'_{yx} = T_{yx}\cos^2\theta - T_{xy}\sin^2\theta - (T_{xx} - T_{yy})\sin\theta\cos\theta, \end{aligned} \tag{108c}$$

where $T'_{\alpha\beta}$ are the components of 3-D Maxwell stress tensor in frame $\bar{\Sigma}'$ related to those in frame $\bar{\Sigma}$. Equations (108a) and (108b) explicitly demonstrate that the electromagnetic field energy density, Poynting vector and momentum density are Lorentz invariant between frames $\bar{\Sigma}$ and $\bar{\Sigma}'$.

Furthermore, as a symmetric tensor, trace of energy-momentum tensor $\bar{T}^{\alpha\beta}$ must be zero:

$$Tr(\bar{T}^{\alpha\beta}) = \bar{T}^{00} + \bar{T}^{11} + \bar{T}^{22} + \bar{T}^{33} = \mu_\circ u_{em} + \left(\vec{B}^2 - \frac{3}{2}\vec{B}^2\right) - \frac{i^2}{c^2}\left(\vec{E}^2 - \frac{3}{2}\vec{E}^2\right) = \mu_\circ u_{em} - \mu_\circ u_{em} = 0, \tag{109}$$

Likewise, the spatial clockwise rotation $\bar{T}^{\alpha\beta} = \tilde{R}_z(\theta)\bar{T}'^{\alpha\beta}R_z(\theta)$ of (x, y) plane about fixed z-axis yields components of the inverse symmetric energy-momentum tensor in frame $\bar{\Sigma}$.

Since $\bar{T}^{\alpha\beta}$ is derived from average sum of the tensor product of covariant and transpose of contravariant (vice versa) and of their duals, it is imperative to confirm that, and by back substitution we should be able to determine the Lorentz invariant electric and magnetic fields between frames $\bar{\Sigma}$ and $\bar{\Sigma}'$. Matching both sides of Eq. (107) and of its inverse, which is not written here to save space, for $\theta = 0^\circ$ one obtains the following matrix equations for the Cartesian components of the electric and magnetic fields in frame $\bar{\Sigma}'(\bar{\Sigma})$ those in frame $\bar{\Sigma}(\bar{\Sigma}')$,

$$\begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}, \tag{110a}$$

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}, \tag{110b}$$

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix}, \tag{110c}$$

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix}, \tag{110d}$$

which are identical to Eqs. (54a), (54b), (54c), and (54d) for $\theta = 0^\circ$. We can then conclude that $\bar{T}^{\alpha\beta}$ can reliably be used to find Lorentz invariant electric and magnetic field according to $\bar{T}'^{\alpha\beta} = R_z(\theta)\bar{T}^{\alpha\beta}\tilde{R}_z(\theta)$ transformation for spatial rotation of (x, y) plane about the fixed z- axis, between two frames. This confirms the Lorentz invariance of symmetric electromagnetic field energy-momentum tensor between two massive inertial frames $\bar{\Sigma}$ and $\bar{\Sigma}'$.

Since Eq. (78) is representing the classical symmetric energy- momentum tensor and is described by using the 4-dimensional relativistic analogue of 3-dimensional Maxwell's stress tensor [4], it is also imperative to see whether it also confirms the Lorentz invariance and trace characteristics described for the symmetric electromagnetic energy-momentum tensor in Eq. (107). This is expected since $\Theta'^{\alpha\beta}$ is also a symmetric tensor just like $\bar{T}'^{\alpha\beta}$. In the framework of the classical Faraday field transformation, Lorentz invariance of the symmetric electromagnetic energy-momentum tensor $\Theta'^{\alpha\beta} = g^{\alpha\beta}\Theta^{\alpha\beta}\tilde{g}^{\alpha\beta}$ in the massive inertial frame $\bar{\Sigma}'$ is written as

$$\Theta'^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_{em} & S_x/c & S_y/c & S_z/c \\ S_x/c & -T_{xx} & -T_{xy} & -T_{xz} \\ S_y/c & -T_{yx} & -T_{yy} & -T_{yz} \\ S_z/c & -T_{zx} & -T_{zy} & -T_{zz} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{111}$$

$$u'_{em} = u_{em}, \quad S'_x = -S_x, \quad S'_y = -S_y, \quad S'_z = -S_z, \tag{112a}$$

$$g'_x = \frac{S'_x}{c^2} = -\frac{S_y}{c^2}, \quad g'_y = \frac{S'_y}{c^2} = -\frac{S_y}{c^2} \qquad g'_z = \frac{1}{c^2}S'_z = -\frac{1}{c^2}S_z, \tag{112b}$$

$$\left.\begin{array}{l} Tr(\Theta'^{\alpha\beta}) = u'_{em} - T'_{xx} - T'_{yy} - T'_{zz} = u'_{em} - u'_{em} = 0 \\ Tr(\Theta^{\alpha\beta}) = u_{em} - T_{xx} - T_{yy} - T_{zz} = u_{em} - u_{em} = 0 \end{array}\right\}, \quad \Rightarrow \quad Tr(\Theta'^{\alpha\beta}) = Tr(\Theta^{\alpha\beta}) \tag{112c}$$

Equations (112a) and (112b) yield invariant electromagnetic energy density, but Poynting vector and linear momentum density turn out to be non-invariant between the frames $\bar{\Sigma}'$ and $\bar{\Sigma}$. This contradicts Lorentz transformation which requires all components of relativistic vector quantities must be invariant. Since $\Theta'^{\alpha\beta}$ is a symmetric tensor, its trace is zero.

However, in the case of trigonometric circular Lorentz boost along the x-axis given by Eq. (48), applying $\Theta'^{\alpha\beta} = L_z(\theta)\Theta^{\alpha\beta}\tilde{L}_z(\theta)$ in Eq. (78) one obtains symmetric tensor in frame $\bar{\Sigma}'$

$$\Theta'^{\alpha\beta} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{em} & S_x/c & S_y/c & S_z/c \\ S_x/c & -T_{xx} & -T_{xy} & -T_{xz} \\ S_y/c & -T_{yx} & -T_{yy} & -T_{yz} \\ S_z/c & -T_{zx} & -T_{zy} & -T_{zz} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{113}$$

For $\theta = 0^\circ$ symmetric energy-momentum tensor $\Theta'^{\alpha\beta}(0) = L_z(0)\Theta^{\alpha\beta}\tilde{L}_z(0)$ in Eq. (113) yields

$$u'_{em} = u_{em}, \quad S'_x = S_x, \quad S'_y = S_y, \quad S'_z = S_z, \tag{114a}$$

$$g'_x = \frac{S'_x}{c^2} = \frac{S_x}{c^2}, \quad g'_y = \frac{S'_y}{c^2} = \frac{S_y}{c^2}, \quad g'_z = \frac{S'_z}{c^2} = \frac{S_z}{c^2}, \tag{114b}$$

$$Tr(\Theta'^{\alpha\beta}) = \Theta^{00} + \Theta^{11} + \Theta^{22} + \Theta^{33} = u_{em} - T_{xx} - T_{yy} - T_{zz} = u_{em} - u_{em} = 0, \tag{114c}$$

Equations (114a) and (114b) yield invariant energy density, Poynting vector, and momentum density between frames $\bar{\Sigma}'$ and $\bar{\Sigma}$. Furthermore, since $\Theta'^{\alpha\beta}$ is a symmetric tensor, Eq. (114c) predicts that trace of $\Theta'^{\alpha\beta}$ is zero in both frames, confirming that $\Theta'^{\alpha\beta}(0) = L_z(0)\Theta^{\alpha\beta}\tilde{L}_z(0)$ and $\Theta^{\alpha\beta} = \tilde{L}_z(\theta)\Theta'^{\alpha\beta}L_z(\theta)$ are symmetric electromagnetic energy-momentum tensors. Component by component matching both sides of Eq. (113) and of its inverse for $\theta = 0^\circ$, one obtains the matrix equations (110a), (110b), (110c), and (110d) for Cartesian components of the electric and magnetic fields in frame $\bar{\Sigma}'(\bar{\Sigma})$ in terms of those in frame $\bar{\Sigma}(\bar{\Sigma}')$. We conclude that using symmetric energy-momentum tensor is the most reliable way of finding invariant electric and magnetic fields for (i) the spatial rotation of planes about a fixed axis, and (ii) the circular Lorentz boost along the direction of motion and between two reference frames.

10.5. Symmetric Electromagnetic Energy-Momentum Tensor and Angular Momentum

Symmetrical electromagnetic field energy-momentum tensor is needed when we consider the conservation of angular momentum of the electromagnetic field [4]. We can write the angular momentum density of electromagnetic field in the following integral and tensor form [4]

$$\vec{L} = \frac{1}{c}\int \vec{x} \times \left(\vec{E} \times \vec{B}\right) dv \qquad \Rightarrow \quad \bar{M}^{\alpha\beta\gamma} = \bar{\Theta}^{\alpha\beta}x^\gamma - \bar{\Theta}^{\alpha\gamma}x^\beta, \tag{115}$$

As pointed out in section 8, classical construction of symmetric electromagnetic energy-momentum tensor $\bar{\Theta}^{\alpha\beta}$ in Eq. (78) is based on the mixed tensors $F^{\alpha}_{\ \beta} = g^{\alpha\sigma}F_{\sigma\beta}$ $(F_\alpha^{\ \beta} = g^{\beta\sigma}F_{\alpha\sigma})$ which has no explicit symmetry characteristics [4]: it is neither symmetric nor asymmetric. Consequently, the symmetric angular momentum tensor must be constructed from the symmetric electromagnetic field energy-momentum tensor $\bar{T}^{\alpha\beta}(\bar{T}^{\alpha\gamma})$, which is defined according to equation (83), as the average sum of the product of covariant (contravariant) and transpose of contravariant covariant) Faraday tensors and of their duals, $\bar{T}^{\alpha\beta} = (F_{\alpha\beta}\tilde{F}^{\alpha\beta} + G_{\alpha\beta}\tilde{G}^{\alpha\beta})/2$. Therefore, symmetric angular momentum density of electromagnetic field must be defined as

$$\bar{M}^{\alpha\beta\gamma} = \bar{T}^{\alpha\beta}x^\gamma - \bar{T}^{\alpha\gamma}x^\beta, \tag{116}$$

Conservation of total angular momentum of electromagnetic field is then defined as

$$\partial_\alpha \bar{M}^{\alpha\beta\gamma} = \left(\partial_\alpha \bar{T}^{\alpha\beta}\right)x^\gamma + \bar{T}^{\gamma\beta} - \left(\partial_\alpha \bar{T}^{\alpha\gamma}\right)x^\beta - \bar{T}^{\beta\gamma} = 0, \tag{117}$$

Since $\bar{T}^{\alpha\beta}$ is symmetric, then first and third terns n Eq. (117) can be eliminated and $\bar{T}^{\gamma\beta} = \bar{T}^{\beta\gamma}$, which proves that conservation of angular momentum of electromagnetic field is symmetric.

10.6. Symmetric Electromagnetic Energy-Momentum Tensor and Einstein Field Equation

The proposed use of Faraday tensor, its dual, and symmetric electromagnetic energy-momentum tensor and its dual can have profound effect in the solution of Einstein equation [19]-[21], written as

$$\bar{G}^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = K\bar{T}^{\alpha\beta}, \qquad \alpha,\beta = 0,1,2,3 \tag{118}$$

where $R^{\alpha\beta}$ is the Ricci curvature tensor with R being the scalar curvature, and $K = 8G/c^4$ is the gravitational constant. In free space $\bar{T}^{\alpha\beta}$ is the fundamental source of the electromagnetic and gravitational fields. Since $\bar{T}^{\alpha\beta}$ is symmetric, then $\bar{G}^{\alpha\beta}$ must be symmetric, so that we can write

$$\begin{pmatrix} \bar{G}^{00} & \bar{G}^{01} & \bar{G}^{02} & \bar{G}^{03} \\ \bar{G}^{10} & \bar{G}^{11} & \bar{G}^{12} & \bar{G}^{13} \\ \bar{G}^{20} & \bar{G}^{21} & \bar{G}^{22} & \bar{G}^{23} \\ \bar{G}^{30} & \bar{G}^{31} & \bar{G}^{32} & \bar{G}^{33} \end{pmatrix} =$$

$$K\begin{pmatrix} \mu_\circ u_{em} & \frac{i}{c}\mu_\circ S_x & \frac{i}{c}\mu_\circ S_y & \frac{i}{c}\mu_\circ S_z \\ \frac{i}{c}\mu_\circ S_x & B_x^2 - \frac{i^2}{c^2}E_x^2 - \frac{1}{2}\left(\vec{B}^2 - \frac{i^2}{c^2}\vec{E}^2\right) & -\frac{i^2}{c^2}E_xE_y + B_xB_y & -\frac{i^2}{c^2}E_xE_z + B_xB_z \\ \frac{i}{c}\mu_\circ S_y & -\frac{i^2}{c^2}E_xE_y + B_xB_y & B_y^2 - \frac{i^2}{c^2}E_y^2 - \frac{1}{2}\left(\vec{B}^2 - \frac{i^2}{c^2}\vec{E}^2\right) & -\frac{i^2}{c^2}E_yE_z + B_yB_z \\ \frac{i}{c}\mu_\circ S_z & -\frac{i^2}{c^2}E_xE_z + B_xB_z & -\frac{i^2}{c^2}E_yE_z + B_yB_z & B_z^2 - \frac{i^2}{c^2}E_z^2 - \frac{1}{2}\left(\vec{B}^2 - \frac{i^2}{c^2}\vec{E}^2\right) \end{pmatrix}, \tag{119}$$

Component by component matching the symmetric tensors in Eq. (119), one can write

$$\begin{aligned}
&\bar{G}^{00} = \bar{R}^{00} = K\bar{T}^{00} = K\mu_\circ u_{em} = (8G/c^4)\mu_\circ u_{em} \\
&\bar{G}^{01} = \bar{G}^{10} = \frac{i}{c}K\mu_\circ S_x, \quad G^{02} = G^{20} = \frac{i}{c}K\mu_\circ S_y, \quad G^{03} = G^{30} = \frac{i}{c}K\mu_\circ S_z \\
&\bar{G}^{11} = KB_x^2 - \frac{i^2}{c^2}KE_x^2 - \frac{1}{2}K\left(\vec{B}^2 - \frac{i^2}{c^2}\vec{E}^2\right), \quad \bar{G}^{12} = -\frac{i^2}{c^2}KE_xE_y + KB_xB_y, \quad \bar{G}^{13} = -\frac{i^2}{c^2}KE_xE_z + KB_xB_z \\
&\bar{G}^{21} = -\frac{i^2}{c^2}KE_xE_y + KB_xB_y, \quad \bar{G}^{22} = KB_y^2 - \frac{i^2}{c^2}KE_y^2 - \frac{1}{2}K\left(\vec{B}^2 - \frac{i^2}{c^2}\vec{E}^2\right), \quad \bar{G}^{23} = -\frac{i^2}{c^2}KE_yE_z + KB_yB_z \\
&\bar{G}^{31} = -\frac{i^2}{c^2}KE_xE_z + KB_xB_z, \quad \bar{G}^{32} = -\frac{i^2}{c^2}KE_yE_z + KB_yB_z, \quad \bar{G}^{33} = KB_z^2 - \frac{i^2}{c^2}KE_z^2 - \frac{1}{2}K\left(\vec{B}^2 - \frac{i^2}{c^2}\vec{E}^2\right)
\end{aligned} \tag{120}$$

Applying the 4-vector differentiation operator $\partial_\alpha = \partial/\partial x^\alpha = (\partial/\partial(ict), \vec{\nabla})$ we can write

$$\begin{pmatrix} -i\partial(K\mu_\circ u_{em})/c\partial t \\ \frac{i}{c}\partial(K\mu_\circ S_x)/\partial x \\ \frac{i}{c}\partial(K\mu_\circ S_y)/\partial y \\ \frac{i}{c}\partial(K\mu_\circ S_z)/\partial z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -i\partial(K\mu_\circ u'_{em})/c\partial t \\ \frac{i}{c}\partial(K\mu_\circ S'_x)/\partial x \\ \frac{i}{c}\partial(K\mu_\circ S'_y)/\partial y \\ \frac{i}{c}\partial(K\mu_\circ S'_z)/\partial z \end{pmatrix} = R_z(-\theta) \begin{pmatrix} -i\partial(K\mu_\circ u'_{em})/c\partial t \\ \frac{i}{c}\partial(K\mu_\circ S'_x)/\partial x \\ \frac{i}{c}\partial(K\mu_\circ S'_y)/\partial y \\ \frac{i}{c}\partial(K\mu_\circ S'_z)/\partial z \end{pmatrix}, \tag{121}$$

Using the chain rule in Eqs. (61) and (62) for space and time differential operators, Eq. (121) can be written as covariant differential form of Poynting theorem with no current source

$$K\mu_\circ \frac{\partial(u_{em})}{\partial t} + K\mu_\circ \vec{\nabla}.(\vec{S}) = \gamma_{tt}^2\left(1-\beta'^2\right)R_z(-\theta)K\mu'_\circ \frac{\partial(u'_{em})}{\partial t'} + \gamma_{x_ix_i}^2\left(1-\beta'^2_{x_i}\right)R_z(-\theta)K\mu'_\circ \vec{\nabla}'.(\vec{S}') \tag{122}$$

Matching both sides of Eq. (120) one finds $\gamma_{x_ix_i} = R_z(\theta)/(1-\beta_{x_i}^2)^{1/2}$ and $\gamma_{tt} = R_z(\theta)/(1-\beta'^2)^{1/2}$ for the spatial rotation dependent space and time components of Lorentz scaling factor, which reduce to those in Eq. (10) without rotation, and transforms covariant Eq. (122) into invariant form in which is the Lorentz invariant classical differential form of Poynting theorem in free space.

It is also important to point out that conservation of electromagnetic energy and linear momentum equations can also be written for the trigonometric circular Lorentz boost along the direction of motion following the steps to write Eq. (122). For a circular boost along the x-axis, we write the following matrix equation for the Poynting theorem under rotational boost

$$\begin{pmatrix} -i\partial(K\mu_\circ u_{em})/c\partial t \\ \dfrac{i}{c}\partial(K\mu_\circ S_x)/\partial x \\ \dfrac{i}{c}\partial(K\mu_\circ S_y)/\partial y \\ \dfrac{i}{c}\partial(K\mu_\circ S_z)/\partial z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -i\partial(K\mu_\circ u'_{em})/c\partial t \\ \dfrac{i}{c}\partial(K\mu_\circ S'_x)/\partial x \\ \dfrac{i}{c}\partial(K\mu_\circ S'_y)/\partial y \\ \dfrac{i}{c}\partial(K\mu_\circ S'_z)/\partial z \end{pmatrix} = L_z(-\theta) \begin{pmatrix} -i\partial(K\mu_\circ u'_{em})/c\partial t \\ \dfrac{i}{c}\partial(K\mu_\circ S'_x)/\partial x \\ \dfrac{i}{c}\partial(K\mu_\circ S'_y)/\partial y \\ \dfrac{i}{c}\partial(K\mu_\circ S'_z)/\partial z \end{pmatrix} \tag{123}$$

from which, similar to writing Eq. (123), one can write the covariant and then invariant form of Poynting theorem with no current source. Steps similar can be taken to write the equation for the conservation of linear momentum in free space.

It is also instructive to demonstrate the Lorentz invariance of the symmetric energy-momentum tensor in the massive inertial frame Σ'. Using Einstein's gravitational field equation (118) we write

$$\bar{T}'^{\alpha\beta} = R(\theta)\bar{T}^{\alpha\beta}\tilde{R}(\theta) \quad \rightleftarrows \quad \bar{G}'^{\alpha\beta} = R(\theta)\bar{G}^{\alpha\beta}\tilde{R}(\theta), \tag{124}$$

Since $\bar{T}^{\alpha\beta}$ is Lorentz invariant, then $\bar{G}^{\alpha\beta}$ must also be invariant between frames $\bar{\Sigma}'$ and $\bar{\Sigma}$. In other words, Einstein field equation must also be Lorentz invariant between frames $\bar{\Sigma}'$ and $\bar{\Sigma}$, written as

$$\bar{G}^{\alpha\beta} - K\bar{T}^{\alpha\beta} = \bar{G}'^{\alpha\beta} - K'\bar{T}'^{\alpha\beta}, \tag{125}$$

This demonstrates the ability of a simple and theoretically reliable derivation of the symmetric energy-momentum tensor and Lorentz invariance of relativistic quantities between two inertial frames in relativistic electrodynamics and in the classical field theory. Knowing the symmetric energy momentum tensor, one can solve Einstein's field equation for the curvature of the universe.

11. Conclusions

We introduced a four-dimensional generalized Minkowski spacetime frame in which the space and time coordinates are linearly interrelated. After satisfying Lorentz invariance of metric equation between two massive inertial frames, we used classical vector transformation to derive general expressions for Cartesian components of the relativistic velocity, which is valid at any speed, including the speed of light. Considering two massive inertial frames form a closed and isolated system in four dimensional spacetime, we integrated the relativistic velocity components with conservation of energy and new way of using Faraday tensor and its dual in Minkowski space-time to prove that the electric and magnetic fields are Lorentz invariant under circular boost and spatial rotation of planes about a fixed axis. We demonstrate that the product of covariant and contravariant Faraday tensors and of their duals lead to non-zero electromagnetic field Lagrange density in free space, which is physically realistic since electromagnetic fields carry energy and momentum while they propagate free space. We derived analytical expressions for Lorentz invariant Maxwell's equations, current continuity equation, and symmetric electromagnetic energy-momentum tensor between two inertial frames, with and without source. We further demonstrated that symmetric energy-momentum tensor can be used have a reliable derivation of Lorentz invariant electromagnetic fields between two inertial frames in the case of circular Lorentz boost and spatial rotation about a fixed coordinate axis. We believe that the proposed theory may have profound effect in creating new research areas in theoretical physics.

Data Availability Statement

This manuscript has no associated data, or the data will not be deposited.

APPENDIX I. Time Dilation, Doppler Shift, and Energy Dispersion Relation

Using Eq. (9) for the general form of four-covariant metric equation under Lorentz transformation, we write

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 = -c^2dt^2\left(1-\frac{u_x^2+u_y^2+u_z^2}{c^2}\right) = -c^2\left(1-\frac{u^2}{c^2}\right)dt^2 \tag{A1a}$$

$$\begin{aligned}\gamma_{\mu\nu}^2 ds'^2 &= \gamma_{xx}^2 dx'^2 + \gamma_{yy}^2 dy'^2 + \gamma_{zz}^2 dz'^2 - c'^2\gamma_{tt}^2 dt'^2 \\ &= -c'^2\gamma_{tt}^2 dt'^2\left[1-\frac{1}{c'^2\gamma_{tt}^2}\left(\gamma_{xx}^2 u_x'^2 + \gamma_{yy}^2 u_y'^2 + \gamma_{zz}^2 u_z'^2\right)\right]\end{aligned} \tag{A1b}$$

where $c'^2 = c^2$, $u^2 = u_x^2 + u_y^2 + u_z^2$, $u'^2 = u_x'^2 + u_y'^2 + u_z'^2$. Matching of Eqs. (A1a) and (A1b) yields

$$\Delta t = \gamma_{tt}\left(1-\frac{u^2}{c^2}\right)^{-1/2}\left(1-\frac{1}{c^2\gamma_{tt}^2}\left(\gamma_{xx}^2 u_x'^2 + \gamma_{yy}^2 u_y'^2 + \gamma_{zz}^2 u_z'^2\right)\right)^{1/2}\Delta t', \tag{A2}$$

For motion along x axis, $u^2/c^2 = u_x'^2/c^2$ and $\gamma_{xx} = \gamma_{tt} = 1/(1-\beta^2)^{1/2}$, Eq. (A2) reduces to

$$\Delta t = \gamma_{tt}\left(1-\frac{u^2}{c^2}\right)^{-1/2}\left(1-\frac{u^2}{c^2}\right)^{1/2}\Delta t' = \left(1-\beta^2\right)^{-1/2}\Delta t' = \left(1-\frac{v^2}{c^2}\right)^{-1/2}\Delta t' = \gamma_{tt}\Delta t', \tag{A3}$$

Using $\omega = 2\pi/T$, Eq. (A3) allows us to write the following expression for the relativistic invariance of Doppler shift between two massive inertial frames under Lorentz transformation

$$\Delta\omega' = \gamma_{tt}\left(1-\frac{u^2}{c^2}\right)^{-1/2}\left(1-\frac{1}{c^2\gamma_{tt}^2}\left(\gamma_{xx}^2 u_x'^2 + \gamma_{yy}^2 u_y'^2 + \gamma_{zz}^2 u_z'^2\right)\right)^{1/2}\Delta\omega, \tag{A4}$$

For motion along the +x axis, $u^2/c^2 = u_x'^2/c^2$ and $\gamma_{xx} = \gamma_{tt} = 1/(1-\beta^2)^{1/2} = \gamma$. Eq. (A4) reduces to $\Delta\omega' = \gamma\Delta\omega$ for the relativistic Doppler shift in the classical four-dimensional spacetime theory.

Recall that Ives and Stilwell [22] who observed the wavelength of hydrogen atom emitted by canal rays with and against their motion by using a mirror and discovered the frequencies of displaced lines of incoming and outgoing light rays and their average are given by [23]

$$\omega_+' = \gamma(1-\beta)\omega; \qquad \omega_-' = \gamma(1+\beta)\omega; \qquad \omega_{av}' = \frac{\omega_+' + \omega_-'}{2} = \gamma\omega \tag{A5}$$

Consider forward and inverse (incoming and outgoing) plane waves of frequencies ω, ω' and wave vectors $\vec{k}(\vec{k}')$ in the massive inertial frames Σ and Σ', respectively, with wave functions

$$\varphi = Ae^{\omega.t\mp\vec{k}\cdot\vec{r}}; \qquad \varphi' = A'e^{\omega'.t'\mp\vec{k}'\cdot\vec{r}'} \tag{A6}$$

where $\omega/k = c$ and $\omega'/k' = c'$ in the massive inertial frames Σ and Σ', respectively. With Lorentz invariant phases of plane waves $\Delta\phi' = 0$ $(\Delta\phi = 0)$ between Σ' and Σ_0 (Σ and Σ_0), we can write

$$\Delta\omega_0 t \mp \vec{k}_0\cdot\vec{r}_0 = \Delta\omega' t' \mp \vec{k}'\cdot\vec{r}'; \qquad \Delta\omega_0 t \mp \vec{k}_0\cdot\vec{r}_0 = \Delta\omega t \mp \vec{k}\cdot\vec{r} \tag{A7}$$

where $\Delta\omega_o$ is the incremental shift in the angular frequencies of plane waves in stationary inertial frame Σ_0. $\Delta\omega'$ and $\Delta\omega$ are the incremental shifts in ω' and ω in frames Σ' and Σ. Using the generalized time $t' = t'(x,y,z,t)$ and velocity, equalities in Eq. (A7) lead to

$$\Delta\omega_0(1-\beta)t = \Delta\omega_+'\gamma_{tt}\left(1-\beta^2\right)t, \qquad \Delta\omega(1+\beta)t = \Delta\omega_-'\gamma_{tt}\left(1-\beta^2\right)t, \tag{A8a}$$

$$\Delta\omega_0(1-\beta)t = \Delta\omega_+\gamma_{tt}\left(1-\beta^2\right)t, \qquad \Delta\omega_0(1+\beta)t = \Delta\omega_-\gamma_{tt}\left(1-\beta^2\right)t, \tag{A8b}$$

where $\vec{k}\cdot\vec{r} = 0$ and $\vec{k}'\cdot\vec{r}' = 0$, which can be proven by using $\omega/k = c$ and $\omega'/k' = c'$, and spacetime coordinate equations. Doppler shifts in the angular frequencies of forward and inverse plane waves and their averages in the massive inertial frames Σ' and Σ are then written as

$$\Delta\omega'_{+} = \frac{(1-\beta)}{\gamma_{tt}\left(1-\beta^{2}\right)}\Delta\omega_{0}, \quad \Delta\omega'_{-} = \frac{(1+\beta)}{\gamma_{tt}\left(1-\beta^{2}\right)}\Delta\omega_{0} \quad \Rightarrow \quad \Delta\omega'_{av} = \frac{\Delta\omega'_{+}+\Delta\omega'_{-}}{2} = \gamma_{tt}\Delta\omega_{0}, \tag{A9a}$$

$$\Delta\omega_{+} = \frac{(1-\beta)}{\gamma_{tt}\left(1-\beta^{2}\right)}\Delta\omega_{0}, \quad \Delta\omega_{-} = \frac{(1+\beta)}{\gamma_{tt}\left(1-\beta^{2}\right)}\Delta\omega_{0} \quad \Rightarrow \quad \Delta\omega_{av} = \frac{\Delta\omega_{+}+\Delta\omega_{-}}{2} = \gamma_{tt}\Delta\omega_{0}, \tag{A9b}$$

Expressions for angular frequencies in the massive inertial frames Σ' and Σ are then written as

$$\omega' = \omega'_{0} + \frac{1}{2}\omega'_{av} = \omega'_{0} + \frac{\Delta\omega'_{0}}{2\sqrt{1-u'^{2}/c'^{2}}}; \qquad \omega = \omega_{0} + \frac{1}{2}\omega_{av} = \omega_{0} + \frac{\Delta\omega_{0}}{2\sqrt{1-u^{2}/c^{2}}} \tag{A10}$$

where ω'_0 and ω_0 are the background angular frequencies and $\Delta\omega'_0$ and $\Delta\omega_0$ are the Doppler shifts with $\Delta\omega'_0/\Delta k' = u'$ and $\Delta\omega_0/\Delta k = u$. Multiplying ω' and ω with $\hbar = h/2\pi$ and using $\hbar\Delta\omega'_0 = (\hbar\Delta k'_0)u' = \Delta p'_0 u' = m'_0 u'^2$ and $\hbar\Delta\omega_0 = (\hbar\Delta k_0)u = \Delta p_0 u = m_0 u^2$, one can write

$$E' = m'c'^{2} = m'_{0}c'^{2} + \frac{m'_{0}u'^{2}}{2\sqrt{1-u'^{2}/c'^{2}}}; \qquad E = mc^{2} = m_{0}c^{2} + \frac{m_{0}u^{2}}{2\sqrt{1-u^{2}/c^{2}}} \tag{A11}$$

where $m_0 c'^2 = \hbar\omega'_0 = E'_0$ and $m_0 c^2 = \hbar\omega_0 = E_0$ are the rest energies of a particle in Σ' and Σ .

APPENDIX II. Relativistic Mass and Energy Dispersion Relation

Expressions for relativistic mass and energy are derived by considering the differential change in the energy of a particle moving under the influence of a force in frames Σ' and Σ as [10], [11]

$$dE' = (\vec{F}'.\vec{u}')dt' = \vec{u}'\cdot d(m'\vec{u}') = \vec{u}'\cdot d\vec{p}' = \frac{1}{m'}\vec{p}'\cdot d\vec{p}' = c^{2}dm', \tag{A12a}$$

$$dE = (\vec{F}.\vec{u})dt = \vec{u}\cdot d(m\vec{u}) = \vec{u}\cdot d\vec{p} = \frac{1}{m}\vec{p}\cdot d\vec{p} = c^{2}dm, \tag{A12b}$$

Using the change of variables, the integrals of expressions in Eq. (A12a) and (A12) are written as

$$\int_{m'(0)}^{m'(u')}\frac{dm'}{m'} = \int_{u'(0)}^{u'}\frac{du'}{c^{2}-u'^{2}}; \qquad \int_{m(0)}^{m(u)}\frac{dm}{m} = \int_{u(0)}^{u}\frac{du}{c^{2}-u^{2}}, \tag{A13}$$

where $u^2 = u_x^2 + u_y^2 + u_z^2$ with $(u(0) = 0)$ and $\sqrt{c^2 - u^2} = \eta$ in the massive inertial frame Σ and $u'^2 = u'^2_x + u'^2_y + u'^2_z$ with $u'(0) = 0$ and $\sqrt{c'^2 - u'^2} = \eta'$ in the massive inertial frame Σ' . The result integrals in Eq. (A13) give the relativistic masses in the massive inertial frames Σ' and Σ

$$m'(u') = \frac{m'(0)}{\sqrt{1-u'^{2}/c^{2}}}, \qquad m(u) = \frac{m(0)}{\sqrt{1-u^{2}/c^{2}}}, \tag{A14}$$

where $m'(0) = m(0) = m_0$ and $u'(0) = u(0) = 0$ are rest mass and initial velocities in both frames.

Since $u^2 = u'^2$, the relativistic mass is Lorentz scalar ($m'(u') = m(u)$) in both frames.

The relativistic energy dispersion relations for a particle moving under the influence of a force in the Σ' and Σ frames are found from the integrals of Eqs. (A12a) and (A12b) that are written as

$$\int_{p'(0)}^{p'(u')} p'dp' = c^{2}\int_{m'(0)}^{m'(u')} m'dm' \quad or \quad \frac{1}{m'}\int_{p'(0)}^{p'(u')} p'dp' = c^{2}\int_{m'(0)}^{m'(u')} dm' \tag{A15a}$$

$$\int_{p(0)}^{p(u)} pdp = c^{2}\int_{m(0)}^{m(u)} mdm \quad or \quad \frac{1}{m}\int_{p(0)}^{p(u)} pdp = c^{2}\int_{m(0)}^{m(u)} dm, \tag{A15b}$$

where $p'(0) = m_0 u'(0) = 0$, $p(0) = m_0 u(0) = 0$ and $p'(u') = m'u'$, and $p(u) = mu$, respectively.

Evaluating the first integrals in Eqs. (A15a) and A15b), then multiply both sides by c'^2 and c^2, and finally taking square root of the final results, one finds Einstein energy dispersion relation

$$E' = \hbar\omega' = m'c'^2 = \left(c'^2 p'^2 + m_0^2 c'^4\right)^{1/2}; \qquad E = \hbar\omega = mc^2 = \left(c^2 p^2 + m_0^2 c^4\right)^{1/2}, \tag{A16}$$

Furthermore, evaluating the second integrals in Eqs (A15a) and (A15b), one can also write

$$E' = \hbar\omega' = m'c^2 = \frac{m_0 u'^2}{2\sqrt{1 - u'^2 / c^2}} + m_0 c^2; \qquad E = \hbar\omega = mc^2 = \frac{m_0 u^2}{2\sqrt{1 - u^2 / c^2}} + m_0 c^2, \tag{A17}$$

Since by $c'^2 = c^2$, $p'^2 = p^2$, and $u^2 = u'^2$, relativistic energy is Lorentz scalar ($E'(u') = E(u)$, which suggests that vector transformation does not affect the relativistic invariance of energy.

Dividing both sides of Eqs. (A16) and (A17) with c'^2 and c^2 ,we and write

$$m_0 = \left(\frac{u'^2 / c'^2}{1 - u'^2 / c'^2} + 1\right)^{-1/2} \frac{\hbar\omega'}{c'^2}, \qquad m_0 = \left(\frac{u^2 / c^2}{1 - u^2 / c^2} + 1\right)^{-1/2} \frac{\hbar\omega}{c^2}, \tag{A18a}$$

$$m_0 = \left(\frac{u'^2}{2\sqrt{1 - u'^2 / c'^2}} + 1\right)^{-1} \frac{\hbar\omega'}{c'^2}, \qquad m_0 = \left(\frac{u^2}{2\sqrt{1 - u^2 / c^2}} + 1\right)^{-1} \frac{\hbar\omega}{c^2}, \tag{A18b}$$

which suggest that the rest mass is linear function of frequency at any ϕ, and (**i**) $m_0 = 0$ at speed of light ($v = c$) when nonstationary frame moves parallel to ±x axes ($\phi = 0, \pi$ and $\theta = \pi / 2$) of stationary frame and (**ii**) $m_0 = \hbar\omega / c^2$ at *v=0* as its limiting case at any angle ($0 \le \phi \le 2\pi$ and $0 \le \theta \le \pi$). The first result (*i*) proves that a relativistic particle has zero mass as it moves with the speed of light, frame independent. The energy dispersion relation (A16) becomes equal to

$$E' = \hbar\omega' = m'c'^2 = c'^2 p'^2; \qquad E = \hbar\omega = mc^2 = c^2 p^2, \tag{A19}$$

The dynamic torsion balance experiment of Liu et al [24] yields an upper bound $m_0^{ub} = 1.2 \times 10^{-54} kg$ at $f = 7.41 x10^{-4} Hz$. Equations (A18a) and (A18b) predict $m_0 = 4.34\text{x}10^{-55}$kg suggesting that a particle at rest having a small magnitude but never zero. Comparison shows a good agreement with measurements [24] and astronomical observations [25].

The second result (*ii*) is compared with the prediction of Heisenberg uncertainty principle in rest frame, which yields an upper bound for the rest mass of photon

$$\Delta E \Delta t \ge \hbar \quad \rightarrow \quad m_0^{ub} = \hbar / c^2 T = (h / 2\pi) f / c^2 = \hbar\omega / 2\pi c^2 \tag{A20}$$

where $\Delta E = m_0^{ub} c^2$ is the photon rest energy and $\Delta t = T = 1/f$. Notice that that the prediction of Eqs. (A18a) and (A18b) is 2π times higher than that predicted by Eq. (A20] Equations (A18) and (A18n) give $m_0 = 8.48 \times 10^{-69}$ kg, compared with $m_0^{ub} = 1.35 \times 10^{-69} kg$ due to Eq. (A20) at $f_0 \approx 2.3 \times 10^{-18} Hz$, where $T_0 = 1/f_0 \approx 13.80\, G$ year is the estimated age of the universe [26].

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