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The Shifted Alliance System of Last Nim Game

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Abstract We introduce a class of impartial combinatorial game which is the multi-player Last Nim game, denoted by $MLNim(N,n)$ in which there are N piles of counters which are linearly ordered, the move will be, the n-player will remove any positive integer of counters from the last pile, we will introduce this $MLNim(N,n)$ with Shifted Standard alliance system by 1, denoted by $SSAM^1$ in which each player will prefer winning for another player over himself. The Aim is to determine the game value of the positions of $MLNim(N,n)$ where $N \geq 1$ is the number of piles and $n \geq 3$ is the number of players and we will present the possible $n \leq N$ and determine the game value in this case.

Keywords Shifted, Alliance, MLNim(N,n), Nim and Impartial combinatorial game

1. Introduction

Combinatorial game theory is a part of mathematics science committed to concentrate the ideal procedure in perfect information data games where commonly two players are included. In a 2-player perfect information game two players substitute moves until one of them cannot move at this turn. Among the games of this sorts, as a non-comprehensive rundown, are Nim Bouton [1], Fraenkel and Lorberbom [2], Flammenkamp [3], Holshouser [4], Albert and Nowakowski [5], Liu and Zhao [6], End-Nim (Albert and Nowakowski [7]), and so on. Last Nim with two players presented by Friedman [8] is played with heaps of counters which are straightly requested. The two players alternate removing any position whole number of counters from the last heap. Under normal play, all P-positions of Last Nim with two players are those containing an odd number of heaps containing one counter.

1.1. Multi-Player Combinatorial Game

Amid the most recent couple of years. The theory of 2-player perfect information games has been generally examined. Normally, it is important to sum up however much as could reasonably be expected of the theory of n-player games. In 2-player perfect information games, one can discuss what the result of the diversion ought to be, at the point when every players play it right, i.e. at the point when every player embraces an ideal strategy yet when there are multiple players', it may not well discuss similar thing. For example, it might so happen that one of the players can help any of the players to win, yet at any rate, he himself needs to lose. Along these lines, the result of the game relies upon how alliances are shaped among the players, in past studies a few conceivable outcomes were researched: multi-player without alliance, multi-player with two alliances and multi-player with alliance system.

1.1.1. Multi-Player without Alliance

N-player Nim game has been submitted without alliance by Li [9]. Straffin [10] tried to classify the three player Nim game with somewhat restrictive assumption regarding the behavior of each player. This work investigated by Loeb [11] by introducing the concept of stable alliance (where an alliance member wins) the work done by Propp [12] analyzed the conditions required to allow one player has a winning strategy against the combined force of the other. Cincotti [13] gave an analysis of n-player partisan games.

1.1.2. Multi-Player with Two Alliances

Kelly ([14], [15]) introduced one bonded Nim, denoted by OBN, which is considered as one pile Nim with two alliance and n-player, any player in this will help his alliance to win, under normal play, the player who removes the last counter will win but under misere play, the player who removes the last counter will lose. The general structures of two alliances were introduced by Zhao and Liu [16].

1.1.3. Multi-Player with Alliance System

Krawec ([17], [18]) considered that every player has a fixed set of alliance, i.e. he will arrange according to preference and this alliance system will be known before beginning the game and this alliance system presented by a matrix. Also he improved a method of analyzing the impartial combinatorial game with n player and considered that one of the players play randomly with no strategy,

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El-Seidy, El kady and Nassar [19] studied the multi-player Last Nim with any alliance system and they made a program which solved the game for any alliance system.

1.1.4. Multiple-Player with Alliance System with Pass

Liu and Yang [20] studied the multiple-player Last Nim when the standard alliance matrix adopted in, this matrix we will explain it below, each of n players either removes any positive integer of counters from the last pile or makes a choice 'pass'. Once a 'pass' option is used, the total number s of passes decreases by 1. When all s passes are used, no player may ever 'pass' again. A pass option can be used at any time, up to the penultimate move, but cannot be used at the end of the game, he determined the game value of the game with different number of piles and players.

2. Multiple-Player Last Nim with Alliance

In this section, we will introduce the Multiple-player Last Nim with alliance and we will introduce the alliance system, the standard alliance system and the game value function which help us to get our results.

$$\begin{pmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,n-1} \\ A_{1,0} & A_{1,1} & & A_{1,n-1} \\ \vdots & & \ddots & \vdots \\ A_{n-1,0} & A_{n-1,1} & \cdots & A_{n-1,n-1} \end{pmatrix}$$

where $A_{i,j}$ determines which player is most preferred choice for the player P_i, and the left-most entries being more preferred over the right-most entries.

Definition 2.1. The standard allaince matrix (for brevity SAM) where $A_{i,j} = j$

$$\begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 0 & 1 & & n-1 \\ \vdots & & \ddots & \vdots \\ 0 & 1 & \cdots & n-1 \end{pmatrix}$$

If we adopt SAM then for each $i \in \{0,1,2,\dots,n-1\}$, player P_i prefers P_i over P_{i+1} over… over P_{n-1} over P_0 over … over P_{i-1}.

Definition 2.2. Using above definitions, Krawec [17] introduced a function $g : C_G \times \mathbb{Z}_n \to \mathbb{Z}_n$ (where C_G denotes the set of all impartial combinatorial games, $\mathbb{Z}_n = \{0,1,2,\dots,n-1\}$)

$$g(G,i) = \begin{cases} 0, & G = \emptyset, \\ A_{i,l}, & other\ wise. \end{cases} \quad (1)$$

Such that

$$l = \min\{i \in \mathbb{Z}_n \mid g(G', i+1) + 1 = A_{i,j}\ with\ G' \in Opt(G)\}$$

Using the above equation Liu and Yang [20] considered any position of MLNim(N; n) as $\boldsymbol{p} = (x_1, x_2, \dots, x_N)$ which has two options are $\boldsymbol{p_0} = (x_1, x_2, \dots, x_{N-1})$ and $\boldsymbol{p_m} = \{(x_1, x_2, x_{N-1}, m) | 1 \le m \le x_{N-1}\}$ then the game value function takes the form:

$$g(\boldsymbol{p}) = \begin{cases} 0, & \boldsymbol{p} = \emptyset, \\ \min\{g(\boldsymbol{p_t}) + 1 \mid 1 \le m \le x_N - 1\}, & other\ wise. \end{cases} \quad (2)$$

3. The Shifted Alliance System of Multi-Player Last Nim

We introduce another alliance matrix is called the shifted alliance matrix by r (for brevity $SSAM^1$) in this matrix $A_{i+j} = j + 1$. If $SSAM^1$ is adopted

we have $A_{i+j} = j + 1$ then

$l = \min\{j \in \mathbb{Z}_n \mid g(G', i+1) + 1 = A_{i,j}\} =$
$\min\{j \in \mathbb{Z}_n \mid g(G') + 1 = j + 1\} =$
$\min\{j \in \mathbb{Z}_n \mid g(G') = j\} = \min\{g(G') \mid G' \in Opt(G)\}$.

$$g(G,i) = \begin{cases} 0, & if\ G = \emptyset \\ \min\{g(G') | G' \epsilon\ opt(G)\} + 1, & if\ other\ wise. \end{cases} \quad (3)$$

Remark 3.1

Liu and Yang (2017) [20] generalized the game value function for any game position for $MLNim(N,n)$ as a vector $\boldsymbol{p} = (x_1, x_2, \dots, x_N)$ that can move in two ways. To it, which are $\boldsymbol{p_0} = (x_1, x_2, \dots, x_{N-1})$ and $\boldsymbol{p_m} = \{(x_1, x_2, \dots, x_{N-1}, m) \mid 1 \le m \le x_N - 1\}$ therefore, the game value function will take the following form

$$g(p) = \begin{cases} 0, & if\ \ \boldsymbol{p} = \emptyset, \\ \min\{g(\boldsymbol{p_t}) \mid 1 \le t \le x_N - 1\} + 1, & if\ \ other\ wise. \end{cases}$$

Definition 3.2. the alliance matrix will be shifted by 1 and will take the form:

$$\begin{pmatrix} 1 & 2 & \cdots & n-1 & 0 \\ 1 & 2 & & n-1 & 0 \\ \vdots & & \ddots & \vdots & \\ 1 & 2 & \cdots & n-1 & 0 \end{pmatrix}$$

If we adopt $SSAM^1$ then for each $i \in \{0,1,2,\dots,n-1\}$, player P_i prefers P_{i+1} over P_{i+2} over…over $P_{i+n-1(mod\ n)}$ over P_i.

Theorem 3.3. For MLNim(N,n) and consider any position vector $\boldsymbol{p} = (x_1, x_2, \dots, x_N)$. If $n \ge N + 1$, thus $g(\boldsymbol{p}) = N$ for all $N \ge 1$.

Proof. First for $N = 0$, then $g(\boldsymbol{p_0}) = 0$ because $\boldsymbol{p_0}$ has no any position can move to it. We will continue by induction on $N \ge 1$.

(i) If $N = 1$, by induction on $x_1 \ge 1$, we will prove that $g(x_1) = 1$. If $x_1 = 1$ we have $g(1) = \min\{g(0)\} + 1 = 1$. For $1 \le m < x_1$ suppose that $g(m) = 1$ then $g(x_1) = min[\{g(0)\} \cup \{g(m), 1 \le m < x_1\}] + 1 = \min\{0,1\} + 1 = 0 + 1 = 1$, then for all $x_1 \ge 1$, $g(x_1) = 1$.

(ii) Suppose that for $1 \le N' \le N - 1$, $g(x_1, x_2, \dots, x_{N'}) = N'$. We will take any $N \ge 2$ and by induction on $t \ge 1$ we will prove that $g(\boldsymbol{p_t}) = N$

Base case: $t = 1$. Then $\boldsymbol{p_1}$ can move to only one position $\boldsymbol{p_0} = (x_1, x_2, x_{N-1})$ with $N' = N - 1$. By induction assumption on $N' = N - 1$ we have $g(\boldsymbol{p_0}) = N - 1$, then

$g(\boldsymbol{p_1}) = \min\{g(\boldsymbol{p_0})\} + 1 = \min\{N - 1\} + 1 = N$.

Induction step: $t \geq 2$. For all $1 \leq m < t$ suppose that $(\boldsymbol{p_m}) = N$, then

$$g(\boldsymbol{p_t}) = min[\{g(\boldsymbol{p_0})\} \cup \{g(\boldsymbol{p_m}) \mid 1 \leq m < t\}] + 1$$
$$= \min\{N-1, N\} + 1 = N \quad (4)$$

In Eq. (4), we put $t = x_N$ then $g(\boldsymbol{p}) = N$.

Theorem 3.4. For MLNim(n+d,n) and consider any position vector $\boldsymbol{p} = (x_1, x_2, \dots, x_{n+d}), d \geq 0, n \geq d + 2$ then

$$g(\boldsymbol{p}) = d + \delta = \begin{cases} d, & if \ \ x_n = 1, \\ d+1 & if \ \ x_n > 1. \end{cases} \quad (5)$$

Proof. We will prove that $g(\boldsymbol{p}) = d + \delta$ by induction on $d \geq 0$

(I) If $d = 0$. For $p = (x_1, x_2, \dots, x_n)$ we will prove that $g(\boldsymbol{p}) = \delta$. We have two cases:

(i) If $x_n = 1$. we will prove that $g(\boldsymbol{p}) = 0$. Let $\boldsymbol{p} = (x_1, x_2, \dots, x_{n-1}, 1)$ then $\boldsymbol{p}$ has only one position can move to it, which is $\boldsymbol{p_0}$ which is a vector position of $MLNim\,(N', n)$ such that $N' = n - 1$ thus $n = N' + 1$. Theorem (3.3) gives that $g(\boldsymbol{p_0}) = N' = n - 1$ then

$$g(\boldsymbol{p}) = \min\{g(\boldsymbol{p_0})\} + 1$$
$$= \min\{n-1\} + 1 = n = 0$$

(ii) For $x_n > 1$. We will prove that $(\boldsymbol{p}) = 1$.

we will prove by induction on $t \geq 2$.

If $t = 2$. Then $\boldsymbol{p_2} = (x_1, x_2, \dots, x_{n-1}, 2)$ has two positions can move to it $\boldsymbol{p_0}$ and $\boldsymbol{p_1}$. For $\boldsymbol{p_0} = (x_1, x_2, \dots, x_{n-1})$ which is a position vector of $MLNim(N', n)$, where $N' = n - 1$, then $n = N' + 1$. Theorem (3.3) gives

$$g(\boldsymbol{p_0}) = N' = n - 1.$$

For $\boldsymbol{p_1} = (x_1, x_2, \dots, x_{n-1}, 1)$ which has only one position can move to it, which is $\boldsymbol{p_0}$, then $g(\boldsymbol{p_1}) = \min\{g(\boldsymbol{p_0})\} + 1 = \min\{n - 1\} + 1 = n = 0$. Then

$$g(\boldsymbol{p_2}) = min[\{g(\boldsymbol{p_0})\} \cup \{g(\boldsymbol{p_1})\}] + 1$$
$$= \min\{n-1, 0\} + 1 = 1.$$

Induction step: $t \geq 2$. Assume that $g(\boldsymbol{p_m}) = 1$ for $1 \leq m < t$. Then

$$g(\boldsymbol{p_t}) = min[\{g(\boldsymbol{p_0})\} \cup \{g(\boldsymbol{p_1})\} \cup \{g(\boldsymbol{p_m})\}]$$
$$+1 \text{ in}\{n-1, 0, 1\} + 1 = 1.$$

(II) Induction step: $d \geq 1$. Assume that $g(x_1, x_2, \dots, x_{n+d'}) = d' + \delta$ for $0 \leq d' < d$ and $n \geq d' + 2$. we will prove that $g(\boldsymbol{p_t}) = d + \delta$. We have $\boldsymbol{p} = (x_1 x_2, \dots, x_{n-1}, 1, x_{n+1}, \dots, x_{n+d})$. By induction on $t \geq 1$.

If $t = 1$ then $\boldsymbol{p} = \boldsymbol{p_1}$ has only one position can move to it, which is $\boldsymbol{p_0}$. The induction assumption then $g(\boldsymbol{p_0}) = d' + \delta$ then $g(\boldsymbol{p_1}) = \min\{g(\boldsymbol{p_0})\} + 1 = d + \delta$.

Induction step: $t \geq 2$, for $1 \leq m < t$ suppose that $g(\boldsymbol{p_m}) = d + \delta$. Thus

$$g(\boldsymbol{p_t}) = min[\{g(\boldsymbol{p_0})\} \cup \{g(\boldsymbol{p_m})\}] + 1$$
$$= \min\{d - 1 + \delta, d + \delta\} + 1 = d + \delta \quad (6)$$

In Eq.(6), we put $t = x_N$, then $g(\boldsymbol{p}) = d + \delta$.

Theorem 3.5. For MLNim(N,n) and consider any position vector $\boldsymbol{p} = (x_1, x_2, \dots, x_{n+d})$. If $n = d + 1 \geq 3$, thus,

$$g(\boldsymbol{p}) = \begin{cases} d & if \quad\quad x_n = 1, \\ d+1 & if\ x_n > 1\ and\ x_{n+d} = 1, \\ d+2 & if\ x_n > 1\ and\ x_{n+d} > 1. \end{cases} \quad (7)$$

Proof.

(i) $x_n > 1\ and\ x_{n+d} = 1$, then $\boldsymbol{p} = \boldsymbol{p_1} = (x_1, x_2, \dots, x_{n+d-1}, 1)$ has only one option $\boldsymbol{p_0} = (x_1, x_2, \dots, x_{n+d-1})$, by letting $d' = d - 1 \geq 1$, we have $n = d + 1 = d' + 2 \geq 3$, then by theorem (3.4) we have $g(\boldsymbol{p}_0) = d' + 1 = d$, thus $g(\boldsymbol{p_1}) = \min\{g(\boldsymbol{p_0})\} + 1 = d + 1$ Hence $g(\boldsymbol{p}) = d + 1$.

(ii) If $x_n > 1\ and\ x_{n+d} > 1$. By induction on $t \geq 2$.
Base case: $t = 2$. Now $\boldsymbol{p_2}$ has two options $\boldsymbol{p_0}$ and $\boldsymbol{p_1}$, from (i) we have $g(\boldsymbol{p_0}) = d$ and $g(\boldsymbol{p_1}) = d + 1$ then

$$g(\boldsymbol{p_2}) = \min[\{g(\boldsymbol{p_0})\} \cup \{\boldsymbol{g}(\boldsymbol{p_1})\}] + 1$$
$$= \min\{d, d+1\} + 1$$
$$= \min\{d, n\} + 1 = \min\{d, 0\} + 1$$
$$= 1 = d + 2$$

(iii) If $x_n = 1$. We will prove that $g(\boldsymbol{p}) = d$ by induction on $t \geq 1$.

Base case: $t = 1$, $\boldsymbol{p_1}$ has only one option $\boldsymbol{p_0}$ of $MLNim(n + d', n)$ where $d' = d - 1$ then $n = d + 1 = d' + 2$, by theorem (3.4) $g(\boldsymbol{p_0}) = d' = d - 1$ then

$$g(\boldsymbol{p_1}) = \min\{g(\boldsymbol{p_0})\} + 1 = d - 1 + 1 = d$$

Induction step: $t > 1$. Suppose that $g(\boldsymbol{p_m}) = d$ such that $1 \leq m < t$ then

$$g(\boldsymbol{p_t}) = \min[\{g(\boldsymbol{p_0})\} \cup \{g(\boldsymbol{p_m}) \mid 1 \leq m < t\}] + 1$$
$$= \min\{d-1, d\} + 1 = d - 1 + 1 = d \quad (8)$$

In eq. (8), we put $t = x_{n+d}$, we have $g(\boldsymbol{p}) = d$.

4. Conclusions

In or article we studied the multi-player Last Nim game in the case of the shifted alliance matrix by one this mean that the each player not prefer himself to win but he prefers the next player who plays after him and we studied the game value function in different cases:

- If the $n \geq N + 1$, where n is the number of players and N is the number of piles we proved that $g(\boldsymbol{p}) = N$. This means that the game value function equal the number of piles.
- If $n \geq d + 2$ where $d = N - n$ we proved that
$$g(\boldsymbol{p}) = d + \delta = \begin{cases} d, & if \ \ x_n = 1, \\ d+1 & if \ \ x_n > 1. \end{cases}$$

5. Future Work

All results given by our paper is based on the assumption that the shifted standard alliance matrix by 1 is adopted and we considered our game without pass.

In the future we will do the following:

(i) apply the shifted alliance matrix in case of MLNim(N,n) with pass.

(ii) We will study the MLNim(N,n) for the shifted alliance matrix by r for $1 < r \leq n-1$.

(iii) we will apply the shifted alliance matrix on different games like Small Nim, Subtraction, games, Wythoff's game, etc.

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