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The Fuzzy Prenucleolus III. The Properties and Monotonic Covers

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Abstract For possessing with prenucleolies fuzzy games we have proved that their sets of prenucleolies can be represented as super differentials. The latter one shows that these two concepts closely affiliated D. Schmiedler in classical theory proved property of super linearity for the nucleolus. We by using theory of superdifferetials the same property extended to fuzzy prenucleolies. For the fuzzy prenucleolies we as well proved property of nonmonotonicity by Megiddo. From geometric point of view, we described the way prenucleolies behave during the process of their construction. Proved that the prenucleolies invariant under operation of monotonic covering. The integration of geometric description for the construction of prenucleolies provides a novel perspective. In our view offers novel insight as well as the proof of invariance of fuzzy prenucleolies for monotonic covers of fuzzy games.

Keywords Fuzzy prenucleolus, Fuzzy prenucleoli

1. Introduction

In classical theory refer to excess based solutions the nucleolus and several other types of nucleolies. Still in nineteen forties game theorists in RAND recommended lexicographic minimization of objective functions. The recommendation published in (Brown G, 1950) and in (Dresher M., 1961).

In (Schmeidler D. 1969) introduced the nucleolus of TU Games. After Schmeidler D., E. Kohlberg in (1971) has described new properties of nucleolus for TU Games. In N. Megiddo (1974) has proved nonmonotonicity of nucleolus. L. Zhu, (1991) has proved the weak coalitional monotonicity of the decision rule. Zhu's result has been extended to fuzzy games in Y. Maroutian (2024).

Section 2 describes the background and current model. Devoted to prenucleolies Section 3. As superdifferential represented sets X^k of prenucleolies. Interpreted geometrically the prenucleolus's behavior while it is in the process of construction and described structure of sets X^k of prenucleolies.

We prove the following properties for games to prenucleolies mappings ν(.), the superlinearity in section 4.1 and Section 4.2 devoted to proving of nonmonotonicity by Megiddo for prenucleolies.

In section 5, we have proved that the prenucleolus remains invariant under operation of monotonic covering.

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2. The Background and Current Model

2.1. Some Preliminaries on Prenucleolies

In theory, use of the term prenucleoli has a common meaning and varies depending on type of games to what it applies. Refer to nucleolies in case of coalitional TU games the nucleolus (Schmeidler [1969], Owen [1977]) and prenucleolus (Sobolev [1975]). It has more general meaning in (Mashler et al [1992]), where nucleolies are decision rules that obtain as a result of lexicographic minimization. In (Mashler, Peleg [1977]) defined generalized nucleoli for set valued dynamic systems. Potters and Tij's [1992] introduced matrix nucleoli and established that it is an analogue of Kohlberg's characterization of nucleolus for TU games in terms of balanced sets (E. Kohlberg [1971]).

In fuzzy theory process of prenucleolus's finding when characteristic function of a game is not of affine type may end up with a solution, which does not possess with a unique value. This kind of decision rule we call prenucleoli of a fuzzy TU game following to classical use of that term.

2.2. Definitions and a Decision Rule

We denote by FG and CFG the sets of all fuzzy and fuzzy concave characteristic function games respectively. FC is the sets of all fuzzy coalitions:

τ: $\tau = (\tau_i, \ldots, \tau_n)$, where $0 \le \tau_i \le 1$. Partially can be
$FC = [0,1]^n$

For a fuzzy game (T, ν), where $T \subseteq [0, 1]^n$ is the game's set of fuzzy coalitions, assume

ν: $T \to R^1$ is it's characteristic function. Set of preimputations is a set of vectors that satisfy to condition of efficiency:

$$X(\nu) = \{x \in R^n / \sum_{i \in N} x_i = \nu(1)\}$$

For k=0, 1, … ,p inductively defined sets X^k, where it is assumed that $X^0 = X$ and $T^0 = \varnothing$. The X^{k+1} are following sets:

$$X^{k+1} = \mathrm{argmin}_{x \in X^k} \sup_{\tau \notin T^k} [(e_\nu(\tau, x) - e_0)/\rho(\tau, T^k)].$$

For k = 1, …, p sets T^k are as below:

$$T^k = \{\tau \in T / \langle x, \tau\rangle = \langle y, \tau\rangle \text{ for every } x, y \in X^k\}$$

$e_\nu(\tau, x) = \nu(\tau) - \langle x, \tau\rangle$ is excess of coalition τ from division vector $x \in X$ and

$e_0 = \min_X \max_{\tau \in T} e_\nu(\tau, x)$. With $\rho(\tau, T^k)$ we denote distance between τ and set T^k

$\rho(\tau, T^k) = \inf_{\tau' \in T^k} \rho(\tau, \tau')$, $\rho(\tau, \tau') = \max_\iota |\tau_\iota - \tau_\iota'|$

If started from some number k_0, for $k > k_0$, $T^{k+1} = T^k$, then that in its turn entails the stabilization of corresponding set X^k or otherwise with increase of k sets X^k do not decrease anymore. That kind of stabilization of sets X^k after finite number of minimization steps may happen in case of games with no piece-wice affine characteristic functions.

We call vectors from sets X^k mentioned above prenucleolies in different of unique vectors that are prenucleoluses.

3. The Prenucleolies

3.1. The Prenucleolies as Superdifferentials and Their Structure

Definitions ([13]). A vector $x \in R^n$ is a supergradient of concave function f: $T \to R^1$ at a point

$\tau^0 \in [0, 1]^n$ if $f(\tau^0) \le f(\tau') + \langle x, \tau^0 - \tau'\rangle$ for every $\tau^0 \in [0, 1]^n$.

The set of supergradients of f at τ^0 is the superdifferential of f at τ^0. We accept farther $\nu \in CFG$.

Below we prove that for a piece-vice affine characteristic function ν if $\nu \in CFG$, at each one of minimization steps set X^k of solutions $X^k = \partial v(\tau)$ where $\tau \in T^{k-1}$. The proof based on the fact that when k increases sets X^k decrease monotonic, i.e. $X^k \subset X^m$, if k>m.

Let us now consider MP discussed at some step m for finding of prenucleolies, i.e.

$$\begin{cases} \min \varepsilon \\ \nu(\tau) - \langle x, \tau\rangle - e_0 \le \varepsilon\rho(\tau, T^m), \text{where } \tau \notin T^m, x \in X^{m-1} \end{cases}$$

If at the same time consider MP, which is at step k, for k>m, then because sets $\{X^l\}$ decrease with l increasing, so we will have that there is $x \in X^m$ such that it belongs also to X^k and that way satisfies to MP of step k:

$$\begin{cases} \min \varepsilon \\ \nu(\tau') - \langle x, \tau'\rangle - e_0 \le \varepsilon\rho(\tau', T^k), \text{where } \tau' \notin T^k, x \in X^k \end{cases}$$

As far as k>m and in each one of the following to m steps of minimization magnitudes of excesses for coalitions $\tau' \notin T^k$ decrease compared with the ones that have been at step m, hence

$\nu(\tau') - \langle\tau', x\rangle - e_0 < \nu(\tau) - \langle\tau, x\rangle - e_0$ from where $\nu(\tau') < \nu(\tau) + \langle x, \tau' - \tau\rangle$.

The last inequality means that each $x \in X^k$ is a supergradient of characteristic function $\nu(\tau)$ from game (T, ν) at $\tau \in T^{k-1}$ or $X^k \subset \partial\nu(\tau)$. The opposite inclusion, i.e. $X^k \supset \partial\nu(\tau)$ one can obtain by taking a vector $x \in \partial\nu(\tau)$. The latter one, i.e that really there is $x \in \partial\nu(\tau)$, and hence, that $\partial\nu(\tau) \ne \varnothing$ one can claim based on concavity of ν. We will come to already obtained inclusion's opposite one by reversing the proof above. That results in the following:

Proposition 3.1.1. For a piece-wice affine concave characteristic function ν $X^k = \partial\nu(\tau)$, where $\partial\nu(\tau)$, is superdifferential of ν at $\tau \in T^{k-1}$.

Remark. Set of concave functions intersects with the set of piece-wice affine characteristic functions. While proving proposition 3.1.1 as by product we have obtained no emptiness of sets X^k. Theory of superdifferentials is a convenient tool for assessing sets X^k and researching through them prenucleolies of concave characteristic function games.

Below we prove another presentation for sets X^k.

Proposition 3.1.2. $X^k = \partial\nu(\tau) = \partial e_\nu(\tau, x)$ for a concave ν and $x \in X^k$.

Proof. We need to show only that $\partial\nu(\tau) = \partial e_\nu(\tau, x)$.

First lets prove that $\partial e_\nu(\tau, x) \subset \partial\nu(\tau)$. Because $\nu \in CFG$ then by τ concave is also $e_\nu(\tau, x)$. Hence, $\partial e_\nu(\tau, x) \ne \varnothing$ and there is $x^* \in \partial e_\nu(\tau, x)$ for $\tau \notin T^k$. Then for a coalition $\tau' \notin T^m$, m>k

$$\frac{\nu(\tau') - \langle x, \tau'\rangle - e_0}{\rho(\tau', T^m)} \le \frac{\nu(\tau) - \langle x, \tau\rangle - e_0}{\rho(\tau, T^k)} + \langle x^*, \tau - \tau'\rangle$$

Because $T = [0, 1]^n$ so for $\varepsilon > 0$ there is $\delta > 0$ such that $|\tau - \tau'|_{max} < \delta$, $|\rho(\tau', T^m) - \rho(\tau, T^k)| < \varepsilon$ and then for ε small enough we will have that $\nu(\tau) - \langle x, \tau\rangle - e_0 \le \nu(\tau') - \langle x, \tau'\rangle - e_0 + \langle x^*, \tau - \tau'\rangle\rho(\tau, T^k)$ from where, $\nu(\tau) \le \nu(\tau') + \langle x, \tau - \tau'\rangle + \langle x^*, \tau - \tau'\rangle\rho(\tau, T^k)$ as far as $|\tau' - \tau|_{max}$ can be arbitrarily small, hence can be so as well $\langle x, \tau - \tau'\rangle$, which means that

$$\nu(\tau) \le \nu(\tau') + \langle x^*, \tau - \tau'\rangle$$

i.e. $x^* \in \partial\nu(\tau)$, and from there $\partial e_\nu(\tau, x) \subset \partial\nu(\tau)$.

The other inclusion $\partial\nu(\tau) \subset \partial e_\nu(\tau, x)$ one can obtain by repeating the same arguments in reverse order.

The proved right now together with Proposition 3.1.1. concludes the proof of our statement.

Proposition 3.1.3. Let (T, ν) be a fuzzy game, where ν is a piece-wice affine, concave characteristic function such that $\nu(\tau) = u(\tau) + \alpha$ and $u(\tau)$ is a linear function. Then for $\tau \in T$ exists $x^* \in \mathrm{ext}\,\partial\nu(\tau)$ satisfying to $\nu(\tau) = \langle x^*, \tau\rangle + \alpha$, where ext $\partial\nu(\tau)$ is the set of extreme points of $\partial\nu(\tau)$.

Proof. Let ν is a concave piece-wice affine characteristic function, $\partial\nu(\tau)$ it's superdiffrential at some point $\tau \in T$ and $x \in \partial\nu(\tau)$ is an arbitrary vector.

Below we will prove that for games possessing with either prenucleolus or prenucleoli and for vector $x^* \in \partial\nu(\tau)$ holds true equality $u(\bar{\tau}) = \langle x^*, \bar{\tau}\rangle$ in case if $\partial\nu(\bar{\tau}) \ne \varnothing$.

We assume that $\nu \in CFG$, $\partial\nu(\tau)$ and $x \in \partial\nu(\tau)$ as they are above. Then $\nu(\tau) \le \nu(\tau') + \langle x, \tau - \tau'\rangle$ for arbitrary $\tau \in T$. From there $\nu(\tau) - \nu(\tau') \le \langle x, \tau - \tau'\rangle$. Farther, let denote by $\bar{\tau} = \tau - \tau'$.

If $\overline{\tau}\in T$, we will have that for every $x\in\partial\nu(\tau)$ $u(\overline{\tau}) \leq \langle x, \overline{\tau}\rangle$. We can obtain the same inequality as well based on $X^k = \partial\nu(\overline{\tau})$ for $\overline{\tau}\in T^k$. By Proposition 3.1.2. for the games that we are dealing with

$X^k = \partial\nu(\overline{\tau})$, when $\overline{\tau}\in T^k$. From there statement of Proposition 3.1.3. takes place if for $\overline{\tau}$ exists T^k such that $\overline{\tau}\in T^k$. For x, $y\in X^k$ if $\overline{\tau}\in T^k$ then $\langle x, \overline{\tau}\rangle = \langle y, \overline{\tau}\rangle$, which means that $\langle x, \overline{\tau}\rangle$ = const for all

$x\in\partial\nu(\overline{\tau})$. The same remains true as well for $x^*\in ext\partial\nu(\overline{\tau})$. Hence, for every $x^*\in ext\partial\nu(\overline{\tau})$ $u(\overline{\tau}) \leq \langle x^*, \overline{\tau}\rangle$.

To conclude the proof let assume that for linear function $u(\tau)$ holds true only $u(\overline{\tau}) < \langle x^* \overline{\tau}\rangle$, when $\overline{\tau}\in T^k$ and $x^*\in\partial\nu(\overline{\tau})$ is an arbitrary vector. By the other side because $u(\tau)$ is a linear function, hence exists $z^*\in extX^k$ such that $u(\overline{\tau}) = \langle z^*, \overline{\tau}\rangle$. From there we are obtaining that

$\langle z^*, \overline{\tau}\rangle = u(\overline{\tau}) < \langle x^*, \overline{\tau}\rangle$, which is a contradiction. That proves our statement, i.e. exists $x^*\in ext\partial\nu(\tau)$ such that $u(\tau)=\langle x^*, \tau\rangle$ and from there in its turn also $\nu(\tau)=\langle x^*, \tau\rangle+\alpha$.

Statement 3.1.4. If $\upsilon(\nu)$ is a prenucleolus of game (T, ν) and k is a number such that $X^k\neq\emptyset$, then $\upsilon(\nu)\in extX^k$.

Proof. It is immediate and follows from definitions of sets X^k and $extX^k$.

Statement 3.1.5. $\partial e_\nu(\tau,x)$ is a convex set if there is number k such that $\tau\in T^k$.

Proof. Follows from Proposition 3.1.2. and convexity of sets X^k.

Statement 3.1.6. For a game (T, v) with linear characteristic function ν if

$\tau\in T^k$, $x\in\partial e_\nu(\tau,y)$ and $x'\in X^k$, then $e_\nu(\tau, x')=\langle\tau, x-x'\rangle$.

Proof. $e_\nu(\tau,x')=\nu(\tau) - \langle x', \tau\rangle$. According to Proposition 3.1.3. for $\tau\in T^k$ $\nu(\tau)=\langle x, \tau\rangle$, where $x\in ext\partial\nu(\tau)=extX^k$. From there we have that if $x'\in X^k$, then $e_\nu(\tau, x')=\langle x, \tau\rangle - \langle x', \tau\rangle=\langle x-x', \tau\rangle$.

The following refers to structure of sets X^k for linear characteristic function games.

Proposition ([13]). If $C\subset R^n$ is compact convex set then C=convh(extC). We note by convhA the convh of set A.

This proposition can be applied to sets $\partial\nu(\tau)$, where ν is a strongly bounded game.

Proposition 3.1.7. For set $\theta(\tau)=\{H\in ext\partial\nu(\tau) / \langle H, \tau\rangle= \nu(\tau)\}$ $X_k=convh\cup_\tau \theta(\tau) \subset convh\partial\nu(\tau)$, where $\tau\in T^k$.

Proof. By Proposition 3.1.3., for some $H\in ext\partial\nu(\tau)$, $\nu(\tau)=\langle H, \tau\rangle$ where $\tau\in T^k$. From there follows the first equation, which is above, i.e. $X^k= convh\cup_\tau \theta(\tau)$. Farther, as far as by definition $\Theta(\tau)\subset ext\partial\nu(\tau)$, hence $convh\cup_\tau \theta(\tau) \subset convhext\partial\nu(\tau)$. This is what has been required.

Remark. Based on Proposition ([13]) in formulation of Proposition 3.1.7 for strongly bounded games instead of inclusion sign takes place equality.

3.2. A Geometric Description on the Prenucleolus's Construction

The way sets X^k behave during the process of prenucleolus's constructing allows to describe it geometrically. That description based on theory of superdifferentials and refers to $\nu\in$CFG.

The position of a set X^k defines by position of its supporting hyperplane. Point $argmin_{X^{k-1}} e_\nu(\tau, x)=x_0$ through what passes supporting hyperplane belongs to $e\rho ie_\nu(\tau,x)$ of map $e_\nu(\tau, x):T\times X\to R^1$ and supports $e\rho ie_\nu(\tau,x)$ with each step of minimization. The position of supporting hyperplane changes relative to position of similar set that has been at previous step. On set $X^k=\partial\nu(\tau)$ position's changes and equivalently changes in directions of supergradients containing in $\partial\nu(\tau)$ during constructing of prenucleolus one can get an idea by comparing expressions of supergradients at two consecutive steps of minimization. At the same time takes place change in position of the supporting hyperplane at each step of the prenucleolus's construction. From there, one can say that while constructing a prenucleolus the supporting hyperplane and together with it the entire set X^k oscillate around a certain position. Sets X^k keep oscillating with each step until they reach their final position at the last step. That is when defines position of the prenucleolus. The latter one of course refers to games that possess with unique prenucleolus.

Now let recall again that the present description is about the class of games that possess with superdifferentials, i.e. games with concave characteristic functions. That class intersects with piece-wice affine games. From there though not every concave characteristic function game has a prenucleolus and hence, it is not always that the described oscillations of sets X_k will come to a final state and stop, however described oscillations will take place for all those games that possess with prenucleolies.

4. Properties

4.1. The Superlinearity of Mapping νu (.)

This part together with property of superlinearity of mapping νu: $\nu\to X_\nu^P$ as a function of ν contains as well a result on nonmonotonicity of prenucleolies by Megiddo.

This property of mapping νu: $\nu\to X_\nu^p$ insighted from Schmeidler (1969) where it relates to nucleolus of classical characteristic function. D Schmeidler defined it in terms of lexicongraphic minimization and hence, for proving the superlinearity the methods that used there are different of ours.

Proposition 4.1.1. The set of CFG of all fuzzy concave characteristic function games is a convex and closed cone. For every $\nu\in$CFG $\nu u(\nu)$ is a superlinear mapping.

Proof. That CFG is a convex and closed cone proves directly. That is the reason for omitting it here. Farther, we are going to deal with sets X_ν^P as superdifferentials $\partial\nu(\tau)$. We have proved in Proposition 3.1.2 that for $\tau\in T_\nu^p$, $X_\nu^P= \partial\nu(\tau)$.

Suppose (T, ν_1) and (T, ν_2) are games, where $\nu_1, \nu_2\in$CFG. By definition of superdifferentials for

$\tau\in T_{\nu_1}^{p_1}$: $\partial\nu_1(\tau) = \{x^1 \in X_{\nu_1}^{P_1} / \nu_1(\tau)\leq\nu_1(\tau')+\langle x^1, \tau-\tau'\rangle\}$

and similarly, for

$\tau\in T_{\nu_2}^{p_2}$: $\partial\nu_2(\tau) = \{x^2 \in X_{\nu_2}^{P_2} / \nu_2(\tau)\leq\nu_2(\tau')+\langle x^2, \tau-\tau'\rangle\}$.

Also, $\partial\nu_1(\tau) \oplus \partial\nu_2(\tau) = \{z/z=x^1+x^2, \ x^1\in \partial\nu_1(\tau), x^2\in\partial\nu_2(\tau)\}$.

That means for $z \in \partial\nu_1(\tau) \oplus \partial\nu_2(\tau)$ takes place:

$(\nu_1+\nu_2)(\tau) \leq (\nu_1+\nu_2)(\tau') + \langle z, \tau-\tau' \rangle$, or otherwise, $z \in \partial(\nu_1+\nu_2)$. Hence, $\partial(\nu_1+\nu_2) \supseteq \partial\nu_1 \oplus \partial\nu_2$.

To prove the superlinearity of νu(ν) for our needs we will use relation $\succ_{(v.o.)}$ for vectors from set X ($x \succ_{(v.o.)} y$ for x, $y \in X$ if for all $i \in N$ $x_i > y_i$).

We require that for vectors $z \in \partial(\nu_1+\nu_2)$, $z \succ_{(v.o.)} x + y$, where $x \in \partial\nu_1(.)$and $y \in \partial\nu_2(.)$. Otherwise, from there it would follow that violates condition of $z \in \partial(\nu_1+\nu_2)$. The latter one means that for arbitrary $z \in \partial(\nu_1+\nu_2)$ holds true inequality $z \geq x+y$, which in its turn means that ν u(ν_1)+ ν u(ν_2) $\leq \nu$u($\nu_1+\nu_2$). From there we have νu(.) really possesses with property of superlinearlity.

4.2. The Nonmonotonicity of Fuzzy Prenucleolies by Megiddo

On this property there is the work of N. Megiddo (1974) for the nucleolus of classical characteristic function games. We extend his result to fuzzy prenucleolies.

Let (T, ν) and (T, u) are two fuzzy TU games and $Z^\nu = Z(T, \nu)$, $Z^u = Z(T, u)$ are sets of prenucleolies for games (T, ν) and (T, u) respectively. We denote by B the following collection of coalitions: $B = \{\tau \in T /$ for $\tau, \tau' \in B$ carrτ=carrτ'=N, and $\nu(\tau)-u(\tau)=\alpha>0\}$.

For all $\tau' \notin B$, i.e. |carrτ|<N, $\nu(\tau')=u(\tau')$.

Definiton 4.2.1. We will say that prenucleolies for games (T, ν) and (T, u) are monotonic if for $x \in Z^\nu$, $y \in Z^u$ and arbitrary $\tau \in B$, $x_i > y_i$ where $i \in N$. We call it monotonicity by Megiddo.

Proposition 4.2.1. For fuzzy games (T, ν), when ν∈CFG, its prenucleolies are not monotonic by Megiddo.

Proof. First, it is clear that prenucleolies $Z^\nu = X_k^\nu$ and $Z^u = X_{k'}^u$ are solutions of known MP's for games (T, ν) and (T, u) at some steps k and k′ when each one of these sets get stabilized. We have proved in Proposition 3.1.2 that $X_k^\nu = \partial\nu(\tau)$, $X_{k'}^u = \partial u(\tau')$, where $\tau \in T_k^\nu$ and $\tau' \in T_{k'}^u$. Let consider superdifferential $\partial(\nu-u)(\tau)$ for coalitions τ and τ′, where $\tau \in B$ and $\tau' \notin B$, as well for vectors $x \in Z^\nu$, $y \in Z^u$: $\partial(\nu-u)(\tau) = \{z / (\nu-u)(\tau) \leq (\nu-u)(\tau') + \langle z, \tau-\tau' \rangle\}$.

Because $\tau' \notin B$, hence $(\nu-u)(\tau')=0$ and $(\nu-u)(\tau) \leq \langle z, \tau-\tau' \rangle$

Now we are going to choose coalitions τ and τ′ the following way:

$$\tau_j = \begin{cases} 0 \; for \; j \in N \backslash carr\tau' \\ \varepsilon_j \; for \; j \in carr\tau \cap carr\tau', where \; 0 < \varepsilon_j < \dfrac{1}{2} \end{cases}$$

And

$$\tau'_k = \begin{cases} 2\varepsilon_k \; for \; k \in carr\tau' \\ 1 \; otherwise \end{cases}$$

In case of chosen that way coalitions τ and τ′ the inequality above holds true only for $z <_{(v.o.)} 0$, which means that $x-y <_{(v.o.)} 0$ or $x <_{(v.o.)} y$. From there, $\nu(\nu)$ is not monotonic. The proposition proved.

Remark. The nonmonotonicity of fuzzy prenucleoluses, which corresponds to the case when $|Z^\nu|=|Z^u|=1$, follows directly from proposition we have proved.

5. On the Prenucleolus of Monotonic Covers

Definition 5.1. For a piece-wice affine characteristic function fuzzy game (T, ν) its monotonic cover is a game $(T, \bar{\nu})$ that for a coalition $\tau \in T$ defines as:

$$\bar{\nu}(\sigma) = \max_{\tau \in B\sigma} \nu(\tau), \text{ where } B_\sigma = \{\tau \in T / \tau \leq \sigma\}$$

The preference relation between coalitions $\tau \in T$ we will consider in this section in sense of vector ordering $\prec_{(v.o)}$

Theorem 5.1 For a fuzzy piece-wice affine characteristic function game its prenucleolus invariant under operation of monotonic covering.

Proof. Let (T, ν) is a fuzzy game that mentioned in theorem's formulation. Below is a minimization problem (MP) for (T, ν):

$$\begin{gathered} \min \varepsilon \\ \nu(\tau) - \langle x, \tau \rangle - e_0 < \varepsilon \\ \text{for } \tau \in T \text{ and } x \in X \end{gathered} \tag{1}$$

We have proved in Y. Maroutian (2017) that solution $(\boldsymbol{\varepsilon}_0, X^1)$ for LPP (1′) below is also solution for MP(1).

$$\begin{gathered} \min \varepsilon \\ \nu(\tau^i) - \langle x, \tau^i \rangle - e_0 < \varepsilon \\ \text{Where } \{\tau^i\} \subset \Sigma^k, k = 1, m, x \in X \end{gathered} \tag{1′}$$

$\{\tau^i\}$ here is set of peaks of simplexes $\{\Sigma^k\}_{k=\overline{1,m}}$ that cover the set of T.

It has been proved that a game (T,ν) possesses with prenucleolus when minimization problems below, where $k = \overline{1,p}$ have solutions:

$$\begin{gathered} \min \varepsilon \\ \bar{\nu}(\sigma) - \langle x, \sigma \rangle - e_0 < \varepsilon_0 \, \rho(\sigma, T^{k-1}(\nu)) \text{ for } \sigma \notin T^{k-1}(\nu) \\ x \in X(\nu) \end{gathered}$$

We assume that set $X^k(\nu)$ is the solution of MP above. There will be need for dealing parallelly with similar constructs that refer to either one of the games: $(T, \bar{\nu})$ or (T, ν). Deserves for separate discussion the case when points of maximum of function $\upsilon(\tau)$ for $\tau \in B_\sigma$ are falling out of sets $B_\sigma \cap T^k$. If for $\tau_0 \in B_\sigma \cap T^k$ $\bar{\nu}(\sigma) = \max_{\tau \in B\sigma} \nu(\tau) = \nu(\tau_0)$ then because from $x \in X^{k-1}(\nu)$ it follows that also $x \in X^1(\nu)$ hence the left hand side of the inequality, which contains the utility function in MP for $\bar{\nu}$ becomes negative due to equality to zero of its right hand side. From there to find points of maximum of ν (τ) considering only set $B_\sigma \cap T^{k-1}$ does not suffice. Hence, instead one should consider the entire set B_σ. Let $(\varepsilon_0, X^1(\nu))$ as above is the solution of MP (1) for game (T, ν). We show first that exists solution of MP for (T, ν) such that $X^1(\nu) \subseteq X^1(\bar{\nu})$.

For $\sigma \in T$ $B_\sigma = \bigcup_{j=1}^m B_\sigma^{\Sigma j}$ where, $B_\sigma^{\Sigma j} = \{\tau / \tau \in B_\sigma \cap \Sigma^j\}$. Every set $B_\sigma^{\Sigma j}$ is compact. In its turn, the characteristic function ν(τ) is continuous. From there on each one of sets $B_\sigma^{\Sigma j}$, there is

$\tau^j \in B_\sigma^{\Sigma j}$ such that $\bar{\nu}(\sigma)|_{\Sigma j} = \max_{\{\tau / \tau \in B_{\sigma \cap \Sigma j}\}} \nu(\tau) = \nu(\tau^j)$.

Due to the finite number of points of maximum $\{\tau^j\}$ exists

τ^0 such that $\bar{\nu}(\sigma) = \max_{\tau\leq\sigma}\nu(\tau) = max_{\overline{1,m}}\ \nu(\tau j) = \nu(\tau^0)$.

Based on that

$\bar{\nu}(\sigma) - <\sigma, x> - e_0 = max_{\tau<\sigma}\nu(\tau) - <\sigma, x> - e_0 = \nu(\tau^0) - <\tau^0, x> + <\tau^0 - \sigma, x> - e_0 \leq \nu(\tau^0) - <\tau^0, x> - e_0 < \varepsilon$.

From there for $x\in X^1(\nu)$ and as well for coalitions σ and τ such that $<x, \sigma - \tau> > 0$ holds true

$x\in X^1(\bar{\nu})$, or $X^1(\nu) \subseteq X^1(\bar{\nu})$.

For $\tau\in B_\sigma$ let $X^{1'}(\nu)$ be the following set:

$X^{1'}(\nu) = \{x\in X^1(\nu) \ / < \sigma-\tau, x> > 0$ for pairs of coalitions such that $\tau\leq\sigma\}$.

The set $X^{1'}(\nu) \neq \varnothing$ because $X^{1'}(\nu)\subset X^1(\nu)$ and there are coalitions σ and τ such that $\tau \leq \sigma$. X(ν) refers to the set of all preimputations of ν. From there, the MP:

$$\begin{cases} min\varepsilon \\ \bar{\nu}(\tau) - <x, \tau> - e_0 < \varepsilon \ for \tau\in T \ and \ x\in X(\nu) \end{cases}$$

possesses with solution $X^1(\bar{\nu}) = X^{1'}(\nu) \subset X^1(\nu)$

Farther we describe sets $X^k(\bar{\nu})$ and $T^k(\bar{\nu})$ for arbitrary k through three statements.

Statement 5.1. For a vector $x\in X^k(\nu)$ takes place also $x\in X^k(\bar{\nu})$ if and only if when for pairs of coalitions σ and τ_0 such that $\tau_0\in B_\sigma$ and $\bar{\nu}(\sigma) = \nu(\tau_0)$ holds true $<x, \sigma - \tau_0> > 0$.

Proof. Necessity. Let $x\in X^k(\nu)$. That means for solution of MP on game (T, ν), which is $(\varepsilon_0, X^k(\nu))$ holds true the following inequality:

(S1) $\nu(\tau) - <x, \tau> - e_0 \leq \varepsilon_0\rho(\tau, T^{k-1}(\nu))$, where $\tau\in\Sigma^j \setminus T^{k-1}$.

To state that the same vector $x\in X^k(\nu)$ belongs as well to $X^k(\bar{\nu})$, means requiring that a similar inequality took place for MP, which this time refers to game $(T, \bar{\nu})$ and has solution $(\varepsilon_0, X^k(\bar{\nu}))$:

(S2) $\bar{\nu}(\sigma) - <x, \sigma> - e_0 \leq \varepsilon_0\rho(\sigma, T^{k-1}(\bar{\nu}))$, where $\sigma\in\Sigma^j\setminus T^{k-1}(\bar{\nu})$.

By the other side

$\bar{\nu}(\sigma) - <x, \sigma> - e_0 = \max_{\tau\leq\sigma}\nu(\tau) - <x, \sigma> - e_0 = \nu(\tau_0) - <x, \sigma> - e_0 = \nu(\tau_0) - <x, \tau_0> - e_0 - <x, \sigma - \tau_0> \leq$ where $\tau_0\in B_\sigma$ is a coalition for what $\nu(\tau_0) = \nu(\tau_0) = \max_{\tau\leq\sigma}\nu(\tau)$.

To obtain that vector $x\in X^k(\bar{\nu})$ in the chain above first we should provide the condition (S1), i.e. requiring what is below:

$\leq\nu(\tau_0) - <x, \tau_0> - e_0 \leq \varepsilon_0\rho(\tau_0, T^{k-1}(\nu))$ when $\tau_0\in\Sigma^j\setminus T^{k-1}(\nu)$.

Of what we have had now first follows that really $x\in X^k(\nu)$, and second that it takes place, when $<x, \sigma - \tau_0> > 0$.

Sufficiency. Let $<x, \sigma - \tau_0> > 0$. We need to prove that if $x\in X^k(\nu)$ then it also belongs to

$X^k(\bar{\nu})$: $x\in X^k(\bar{\nu})$. From $x\in X^k(\nu)$ we have that for arbitrary $\tau\in\Sigma^j\setminus T^{k-1}(\nu)$

$\nu(\tau) - <x, \tau> - e_0 \leq \varepsilon_0\rho(\tau, T^{k-1}(\nu))$.

$\bar{\nu}(\sigma) - <x, \sigma> - e_0 = \nu(\tau_0) - <x, \tau_0> - e_0 - <x, \sigma-\tau_0>$, where $\nu(\tau_0) = \max_{\tau\leq\sigma}\nu(\tau)$.

From $<x, \sigma-\tau_0> > 0$ it follows that $\bar{\nu}(\sigma) - <x, \sigma> - e_0 \leq \nu(\tau_0) - <x, \tau_0> - e_0 \leq \varepsilon_0\rho(\tau_0, T^{k-1})$ and hence, $x\in X^k(\bar{\nu})$.

Statement 5.2. For $x\in X^k(\nu)$, $x\notin X^k(\bar{\nu})$ when exists a pair (σ, τ_0) such that $<x, \sigma - \tau_0> \leq - \varepsilon_0\rho(\sigma - \tau_0, T^{k-1})$.

Proof. If for $x\in X^k(\nu)$ $x\notin X^k(\bar{\nu})$ then there is a coalition $\sigma\notin T^{k-1}$ such that

$\bar{\nu}(\sigma) - <x, \sigma> - e_0 > \varepsilon_0\rho(\sigma, T^{k-1})$

$\bar{\nu}(\sigma) - <x, \sigma> - e_0 = \max_{\tau\leq\sigma}\nu(\tau) - <x, \tau_0> - e_0 - <x, \sigma-\tau_0> =$
$= \nu(\tau_0) - <x, \tau_0> - e_0 - <x, \sigma-\tau_0> > \varepsilon_0\rho(\sigma, T^{k-1})$

then because $\nu(\tau) - <x, \tau> - e_0 \leq \varepsilon_0\rho(\tau, T^{k-1})$ for arbitrary $\tau\notin T^{k-1}$ so from there, $- <x, \sigma - \tau_0> \geq \varepsilon_0\rho(\sigma, T^{k-1}) - \varepsilon_0\rho(\tau_0, T^{k-1}) \geq \varepsilon_0\rho(\sigma - \tau_0, T^{k-1})$ by the property of triangle for metrics.

That is what required. As a result we obtain that

$X^k(\bar{\nu}) = \{x\in X^k(\nu) \ / <x, \sigma - \tau_0> \geq 0$ for arbitrary pair of coalitions σ and τ_0 such that $\tau_0 \leq \sigma$ and $\bar{\nu}(\sigma) = \nu(\tau_0)\}$. Hence, $X^k(\bar{\nu}) \subseteq X^k(\nu)$

Remark 5.1. Sets $X^k(\bar{\nu}) \neq \varnothing$, because $X^k(\bar{\nu})\cap X(\nu) \neq \varnothing$, where X(ν) is the set of all preimputations of game (T, ν).

Statement 5.3. For arbitrary k $T^k(\nu) = T^k(\bar{\nu})$.

Proof. Let $(\varepsilon_0, X^k(\bar{\nu}))$ is the solution of k'th MP for game $(T, \bar{\nu})$. That means we can state the following: if $x\in X^k(\bar{\nu})$, then for arbitrary $\tau\notin T^k(\bar{\nu})$, ε_0 is the minimal value of ε in the inequality:

$$\bar{\nu}(\tau) - <x, \tau> - e_0 < \varepsilon\rho(\tau, T^k(\bar{\nu})).$$

From there, because $\bar{\nu}(\tau) = max_{\tau'\leq\tau}\nu(\tau')$ so $\bar{\nu}(\tau) \geq \nu(\tau)$ and $\nu(\tau) - <x, \tau> - e_0 \leq \bar{\nu}(\tau) - <x, \tau> - e_0 < \varepsilon_0\rho(\tau, T^k(\bar{\nu}))$, for arbitrary $x\in X^k(\bar{\nu})\subset X^k(\nu)$ and ε_0. At the same time (ε_0, x) also minimizes the expression $\nu(\tau) - <x, \tau> - e_0$ because $x\in X^k(\nu)$. That means $\nu(\tau) - <x, \tau> - e_0 \leq \varepsilon_0\rho(\tau, T^k(\nu))$ for $\tau\notin T^k(\nu)$.

Farther, for arbitrary $x,y\in X^k(\bar{\nu})$ $<x, \tau> = <y, \tau>$, where $\tau\in T^k(\bar{\nu})$. Because the same hold true for $\tau\in T^k(\nu)$, hence $T^k(\bar{\nu}) \subset T^k(\nu)$.

Let's now move ahead and prove the opposite inclusion, i.e. $T_k^k(\bar{\nu}) \supset T_k^k(\nu)$.

Assume that $\tau\in T^k(\nu) \setminus T^{k-1}(\nu)$. Then for ε_0 and $x\in X^k(\nu)\cap X^k(\bar{\nu})$ holds true the following inequality:

$$\nu(\tau) - <x, \tau> - e_0 \leq \varepsilon_0\rho(\tau, T^{k-1}(\nu)). \quad (*)$$

There is $\tau^0\leq\tau$ such that $\tau^0 \in T^k(\nu)\setminus T^{k-1}(\nu)$ and $\bar{\nu}(\tau) - <x, \tau> - e_0 \leq max_{\tau'\leq\tau^0}\nu(\tau') - <x, \tau> - e_0 \leq \nu(\tau^0) - <x, \tau^0> - e_0 \leq \varepsilon_0\rho(\tau^0, T^{k-1}(\nu))$.

In case, when $\tau^0\in T^{k-1}(\nu)$ (or $\tau^0\in T^i(\nu)$ for some $i \leq k-1$), the last inequality in the above chain remains true, because then $\rho(\tau^0, T^{k-1}(\nu)) = 0$ and from where it's left side is negative. As far as $T^k(\bar{\nu})\subset T^k(\nu)$, so $\tau^0\notin T^{k-1}(\bar{\nu})$ for $\tau^0\notin T^{k-1}(\nu)$. From there, $\rho(\tau^0, T^{k-1}(\nu)) \leq \rho(\tau^0, T^{k-1}(\bar{\nu}))$ for $\tau^0\notin T^{k-1}(\bar{\nu})$.

By definition of metrics $\rho(\tau, T)$ we have that if $\tau^0\leq\tau$, then $\rho(\tau^0, T^{k-1}(\bar{\nu})) = min_{\tau'\in T^{k-1}(\bar{\nu})}\ \rho(\tau^0, \tau') = min_{\tau'\in T^{k-1}(\bar{\nu})}\ \max|\tau_i^0 - \tau_i'| \leq min_{\tau'\in T^{k-1}(\bar{\nu})}\max_i|\tau_I - \tau_i'| = min_{\tau'\in T^{k-1}(\bar{\nu})}\ \rho(\tau, \tau') = \rho(\tau, T^{k-1}(\bar{\nu}))$.

So the inequality $\tau^0\leq\tau$ entails that $\rho(\tau^0, T^{k-1}(\bar{\nu})) \leq \rho(\tau, T^{k-1}(\bar{\nu}))$.

As a conclusion we will have that

$\bar{\nu}(\tau) - <x, \tau> - e_0 \leq \varepsilon_0\rho(\tau^0, T^{k-1}(\bar{\nu})) \leq \varepsilon_0\ \rho(\tau, T^{k-1}(\bar{\nu}))$ or just $\bar{\nu}(\tau) - <x, \tau> - e_0 \leq \varepsilon_0\rho(\tau, T^{k-1}(\bar{\nu}))$, where $\tau\notin T^{k-1}(\bar{\nu})$ (**).

The latter one means (ε_0, x), the pair, which minimizes (**) inequality, does the same also with the inequality (*).

Something that refers this time to MP for game (T, $\bar{v}$).

So, $\tau \in T^k(v) \subset T^{k-1}(\bar{v})$. From there, $T^k(v) \subset T^k(\bar{v})$. The proved right now inclusion with the opposite one, which we have proved above provides as with one required by statement 5.3.

Remark 5.2. Based on equality $T^k(\bar{v}) = T^k(v)$ from statement 5.3. we can claim the following properties for sets $T^k(\bar{v})$:

a.) Sets $T^k(\bar{v})$ are convex

b.) Sets $\{T^k(\bar{v})\}$ are monotonic increasing

c.) With increase of k increases also $\dim T^k(\bar{v})$

Remark 5.3. Sets $X^k(\bar{v})$ are convex.

The proof of this statement is immediate. To prove increasingness of $\dim T^k(\bar{v})$, we don't use the mentioned property of sets $X^k(\bar{v})$. Now we are ready to conclude the theorem's proof. As we mentioned that above the equality $T^k(v)=T^k(\bar{v})$ provides growth of $\dim T^k(\bar{v})$ with each pair of new sets $X^k(\bar{v})$ and $T_k(\bar{v})$ of game (T, $\bar{v}$) that are constructed. From there, after finite number of steps due to convexity of sets $\{T^k(\bar{v})\}$ last set $T^p(\bar{v})$ will coincide with T and the corresponding to it set $X^p(\bar{v})$ will consist of only one vector, which is prenucleolus ℵ($\bar{v}$) of game (T, $\bar{v}$). This results to conclusion of theorem's proof with following statement.

Statement 5.4. If (T, ν) is a piece-wice affine fuzzy characteristic function game and X(ν) is it's set of preimputations then the prenucleolus of monotonic cover (T, $\bar{v}$) of game (T, ν) conincides with prenucleolus of (T, ν), which is ℵ (ν).

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Glossary and List of Notations

$\Gamma = (T, v)$ – fuzzy game with set of coalitions T and characteristic function v. Briefly, we say also game Γ or game v.

$X(v)$ – set of preimutations of game v, which is a set of vectors that satisfy to condition of efficiency: $X(v) = \{x \epsilon R^n \ / \ \Sigma_{i\in N}\, x_i = v(1)\}$.

X^{k+1} – inductively defined sets where k= 0,1,…,p ($p < \infty$). Each one of sets X^k corresponds to k-th step of construction of prenucleolus:

$X^0 = X, X^{k+1} = argmin_{x\in X^{k+1}} max_{\tau\notin T^k}\ [e_v(\tau, x) - e_0)/\rho(\tau, T^k)\}$

T^k – set of coalitions corresponding to k' th step of constructing of prenucleolies: $T^0 = \emptyset$

$e_v\,(\tau, x) = v(\tau) - < x, \tau >$ - excess of coalition τ from division vector x, where

$< x, \tau > = \Sigma_{i\in N}\, x_i\, \tau_i$ is inner product of vectors x and τ.

$e_0 = min_X\, max_{\tau\in T} e_v\,(\tau, x)$.

$\rho(\tau, T^k)$ – distance between coalition τ and set T^k.

$\rho(\tau, T^k) = inf_{\tau' \in T^k}\ \rho(\tau, \tau')$, where $\rho(\tau, \tau') = max_i |\tau_i - \tau'_i|$.

prenucleolus – unique vector to which results process of constructing of sets X^k after finite number of steps. Possess with prenucleolus piece- vice affine games.

Prenucleoles- set of vectors that obtain as a result of stabilization of sets X^k started from some number k_0, i.e. for $k' > k_0$ $X^{k'} = X^{k'+1}$

MP – a minimization problem that discussed at some step m for finding of prenucleolies.

$$\min \varepsilon$$

$v\,(\tau) - < x, \tau > - e_0 \ \le \varepsilon\ \rho(\tau, T^{m-1})$, where $\tau \notin T^{m-1}$ x$\in X^{m-1}$

FG, CFG – sets of fuzzy and fuzzy concave characteristic function games.

FC – set of all fuzzy coalitions: partially FC= $[0,1]^n$

$\nu\,(v)$ – the prenucleolus of characteristic function game (T, v).

$\partial v(\tau)$ and $e_v(\tau, x)$ – superdifferentials respectively of game v (.) at point τ and excess $e_v(\tau, x)$.

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