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Evaluation of the Monopoly as a Stable Second Best

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Abstract When one considers competitive equilibrium with differentiated products (Bertrand competition) it occurs that the Nash equilibrium is unstable. One product should be withdrawn from the market: the asset concerned is bought then closed down, and it is profitable. Therefore, this question is posed to the regulator: what is the second best, the Nash equilibrium with only two products or the monopoly selling three products? In the article, one states that there are three cases: - The monopoly is the second best. One demonstrates that this involves that if the monopoly sells a new product, it creates more consumers ‘surplus. Indeed, a monopoly selling a new product, can create, or not, more consumers' surplus. To give examples of both, is easy. - The second best is the Nash equilibrium (with two products) but the monopoly selling a new product creates more consumers' surplus. - The second best is the Nash equilibrium (with two products) and the monopoly selling a new product does not create more consumers' surplus. The strength of the monopoly is the stability: it keeps the diversity of the products sold. But it chooses high prices. In the first case, the monopoly is the second best because there is a real diversity of the products sold. In the third case, there is a pseudo-diversity of the products sold. The second case is intermediate. The monopoly selling a new product creating more consumers' surplus, is a criterium for the diversity of the products sold.

Keywords Bertrand competition, Regulation, Consumers' surplus

1. Introduction

When one considers competitive equilibrium between differentiated products (Bertrand competition), it occurs that the Nash equilibrium is unstable. That is to say: a spontaneous process is possible, such that one product is withdrawn from the market. The concerned asset is bought then closed down, and the two operations are profitable [1]. A question is posed to the regulator, supposed to be unable to prohibit this process, which is detrimental to the consumers (the prices increase) [1]. This question is: what is the second best he should choose, the monopoly selling three products, or the Nash equilibrium after the withdrawal of a product (only two products are sold).

The article deals with this question.

One demonstrates in the article that there are three situations:

- The second best is the monopoly selling three products. Also, and it is a necessary condition, when the monopoly sells a new product, if generates more consumers' surplus.
- The second best is the Nash equilibrium (two products being sold), but the monopoly selling a new product creates more consumers' surplus.
- The second best is the Nash equilibrium (with two products sold) and the monopoly selling a new product does not create more consumers' surplus.

A multiproduct monopoly has a strength and a weakness. The strength: it keeps the diversity of the products sold. It never cancels the sale of a product, its benefit would decrease. The weakness: it chooses high prices. So, the existence of two cases is explained. Either there is a real diversity of the products sold, and there is more consumers' surplus created when the third product is sold (by the monopoly). Either it is a pseudo-diversity of the products sold, and the Nash equilibrium (only two products being sold) creates more consumers' surplus than the monopoly (selling three products), because of the low prices. The second case is an intermediate one.

It is easy to provide examples of both.

It is also the opportunity to present the method used, Bertrand competition the demands being deduced from the consumers' utilities. Here, in a frame $Ou_1u_2u_3$ ($0 \leq u_i \leq 1$) the density of utility is linear, homogeneous, equal to $1 / \sqrt{2}$, on the bisectors of the faces of the cube $u_i = 0$ (figure 1). If it is the Nash equilibrium, the Bertrand paradox applies, and (with obvious notation): $p_1 = p_2 = p_3 = 0$, $P_1 = P_2 = P_3 = 0$. So, the player E_1 can buy and close E_3, then E_2, paying $\varepsilon / 2$ to each. Now the density is on $O u_1$, equal to 2/3, with a weight 1/3 at O. E_1 chooses $p_1 = ½$, its profit is $1/6 - \varepsilon$, the consumers surplus is 1/12. If the regulator chooses the monopoly, the prices are $p_1 = p_2 = p_3 = ½$, and the consumers' surplus is 1/8. Therefore, the monopoly is the second best.

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But, of course, the first best if the Nash equilibrium with three products sold, since the consumers' surplus is ½ . But it is unstable.

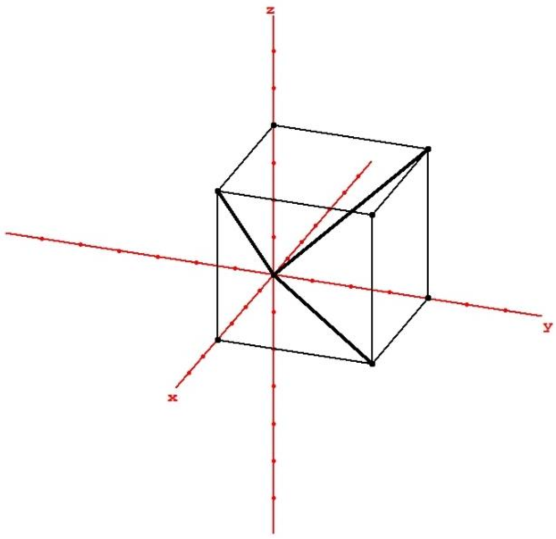

Figure 1. The density of utility is linear homogeneous and on the bisectors of the faces $u_i = 0$

Concerning an example of the second best being the Nash equilibrium (two products being sold), one can quote the example which is in the article of the author [1]. The density of utility is as on figure 2. The Nash equilibrium is unstable. The second best is obviously the Nash equilibrium. It creates a positive consumer*s'* surplus. The monopoly should choose $p_1 = p_2 = p_3 = 1$, and the consumers' surplus would be 0. It is always the case, when the utilities are in the faces of the cube $u_i = 1$.

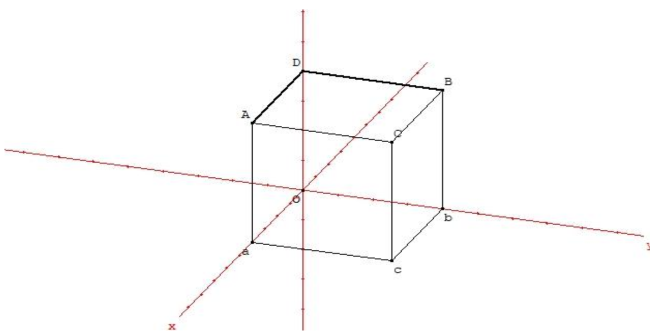

Figure 2. The density of utility is linear homogeneous on (0, 0, 1), (0, 1, 1) and (0, 0, 1), (1, 0, 1).

An interesting question is: does a monopoly selling a new product create more consumers' surplus? The two are possible (it creates or not more consumers' surplus). Here are two examples of both.

If the demands D_1, D_2, D_3 are independent, clearly the monopoly selling a new product creates more consumers' surplus. For instance, the density is linear, homogeneous (equal to 1 / 3) and on the axes O u_i between 0 and 1 (figure 3). When the monopoly selling the products 1 and 2 sells the product 3, the consumers' surplus increases from 1 / 12 to 1 / 8.

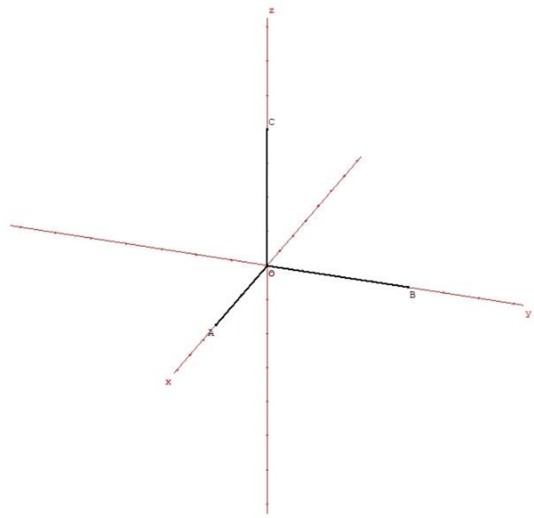

Figure 3. The density of utility is linear, homogeneous on the axes O u_i between 0 and 1

For an example of decreasing consumers' surplus (when a monopoly sells a new product), consider this case: the density is areal, equal to 1 in the plane $u_3 = 1$ (figure 4). When the monopoly sells the products 1 and 2, the price is $p_1 = p_2 = 0{,}57$ (the calculation is made further in the paper). Therefore, the consumers' surplus is positive. When the monopoly sells the product 3, also, it chooses $p_1 = p_2 = p_3 = 1$, and the consumers' surplus is 0.

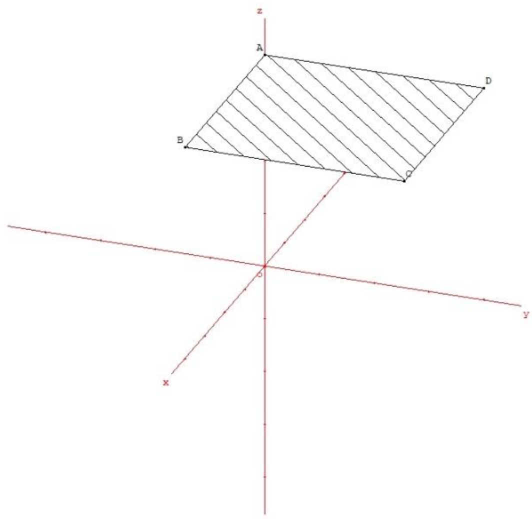

Figure 4. The density is areal, homogeneous, equal to 1, in the plane $u_3 = 1$

If one requires the Nash equilibrium being unstable, to have examples is easy, too.

In the tractable example which is described in the article, the Nash equilibrium is unstable and the monopoly selling a new product creates more consumers' surplus.

For the other case (decreasing consumers' surplus), one can resume the example quoted in the article of the author [2]. The utilities are in the faces of the cube $u_i = 1$. If the equilibrium is stable, non-differentiating innovation makes it unstable when pushed far enough [2]. And the monopoly selling the products 1 and 2 creates consumers' surplus. When it sells the product 3, also, the consumers' surplus is 0 (it chooses (p_0, p_0, p_0), the corner of the cube opposed to (0, 0, 0)).

Now, one can set out the plan of the article:

- Introduction
- The framework. The framework is presented. The main results are set out. The hurried reader could leapfrog the two successive chapters, then read the Conclusion.
- Results and demonstrations.
- A tractable, particular case is presented. It is the simplest case, with a density volumic, homogeneous, and equal to 1.
- Conclusion.

Some demonstrations, which are (a little) awkward, are put in Appendixes.

2. The Framework

In the frame O u_1 u_2 u_3 ($0 \leq u_i \leq 1$) there is a density of utility, the points (u_1, u_2, u_3) representing consumers with these gross utilities. For (p_{10}, p_{20}, p_{30}), D_1 (p_{10}, p_{20}, p_{30}), the demand for the product 1, is the quantity in the "pocket":

$u_1 - p_{10} \geq u_2 - p_{20}$
$u_1 - p_{10} \geq u_3 - p_{30}$
$u_1 - p_{10} \geq 0$.

It corresponds to the consumers buying the product 1 because their net utility is maximized when they buy the product 1. It is obvious that D_i is decreasing in p_i, increasing in p_j and p_k. Also, if CS (p_{10}, p_{20}, p_{30}) is the consumers' surplus, it is straightforward that d CS = $-D_1$ dp_1 $-D_2$ dp_2 – D_3dp_3, so (D_1, D_2, D_3) derives from a potential (-CS) and $\partial D_i / \partial p_j = \partial D_j / \partial p_i$. It is called Slutsky's relation [3]. There are other interesting mathematical relations, such as $\partial D_1 / \partial p_1 + \partial D_2 / \partial p_2 + \partial D_3 / \partial p_3 \leq 0$. Some will be used further in the article.

The functions D_i are supposed to be analytical, two times continuously derivable.

Concerning Bertrand competition, for p_3 fixed, one supposes a best response function R_1 (p_1, p_2) and R_2 (p_1, p_2): if the best response p_{10} is obtained a single time (for p_{20}), R_1 is monotonic, and increasing because it is observed that the price increases from Nash equilibrium to monopoly (corresponding to $p_2 = 1$, that is to say $D_2 = 0$). So, the prices are strategic complements.

One supposes a unique Nash equilibrium, therefore it is stable (in the sense that groping led the players to the equilibrium).

If the density is volumic, $D_i = 0$ if $p_i = 1$, for any p_j, p_k.

One supposes also a symmetry of the utilities about the planes $u_i = u_j$.

Now, one refers to figure 5.

In coordinates, p_3 and in abscissas, it is the bisector of the plane $p_3 = p_{30}$, on which are the equilibrium points, given the symmetry (of the utilities):

- The points such as M_1 represent Nash equilibrium (for p_3 fixed). They are on the curve C_1: M_1, P_1, Q_1, R_1. The slope of C_1 is positive (in other words, the point representing Nash equilibrium slides toward the right when p_3 increases from 0 to 1). M_1, on the bisector $p_1 = p_2 = p_3$ represents the Nash equilibrium with three products.
- The points such as P_2 are the maxima of $P_1 + P_2$ (for p_3 fixed). They are on the curve C_2: P_2, Q_2, R_2. The slope is positive. The points representing the maxima of $P_1 + P_2$ slide toward the right, when $p_3$increases from 0 to 1. P_2 is the Nash equilibrium of E selling the products 1 and 2, and E_3 selling the product 3.
- The points like Q_3 are the maxima of $P_1 + P_2 + P_3$ (for p_3 fixed). They are on the curve C_3: Q_3, R_2 (the maxima of $P_1 + P_2$ and $P_1 + P_2 + P_3$ are the same when $p_3 = 1$ because $D_3 = 0$ and $P_3 = 0$ for $p_3 = 1$). The slope of C_3 is negative. Q_3 is the maximum of $P_1 + P_2 + P_3$ on the bisector $p_1 = p_2 = p_3$.

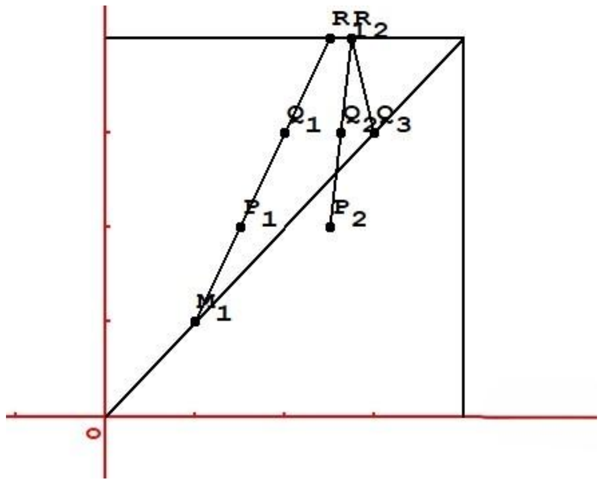

Figure 5. The equilibrium points are shown on the bisector of the plane $p_3 = p_{30}$ when p_{30} varies from 0 to 1. Notice that the sign of the slopes of the curves is shown, but there are not necessarily straight lines

What is interesting is that the slope of C_3 is negative. The consequences are:

- If the monopoly selling three products is the second best (in Q_3) it is not thanks to price dominance (meaning that $p'_i \leq p_i$ for any i).
- If the second best is the monopoly (Q_3, not R_1), if the monopoly selling to products sells a new product, the consumers' surplus increases. Δ CS from R_1 to Q_3 is positive, Δ CS from R_2 to Q_3 is more, therefore it is positive. It is a necessary condition.
- If the Nash equilibrium is the second best (in R_1), Δ CS between R_1 and Q_3 is negative. Possibly, if the monopoly selling two products sells a new product, the consumers' surplus increases (Δ CS is positive between R_2 and Q_3). It is the intermediate case. The tractable example below is thus. Possibly, when the monopoly selling two products sells a new product, it makes the consumers' surplus decrease.

Therefore, the strength and the weakness of the monopoly appear. It keeps the diversity of the products sold, and it is favorable to the consumers. But choosing high prices is unfavorable to them. In case of a real diversity of the products, it creates more consumers' surplus when it sells the product 3 and should be the second best. In case of pseudo-diversity of the products sold, when it sells the product 3, it makes the consumers' surplus decrease. It is sure that the second best is the Nash equilibrium.

And there is the intermediate case: the products are differentiated, but not enough. The monopoly selling the product 3 makes the consumers surplus increase, but there are high prices, so the second best is the Nash equilibrium (with two products sold).

3. Results and Demonstrations

Several demonstrations are needed.

Proof that the curve C_2 is on the right of C_1.

The point of C_2 (choice of the monopoly selling the products 1 and 2, p_3 fixed) is such that $\partial P_1 + P_2 / \partial p_1 = \partial P_1 + P_2 / \partial p_2 = O$. Since $\partial P_1 / \partial p_2 \geq 0$, $\partial P_1 / \partial p_1 \leq 0$, the point is on the right of the point where $\partial P_1 / \partial p_1 = 0$ (on C_1).

Proof that C_3 is on the right of C_2.

A point of C_3 is such that $\partial P_1 + P_2 + P_3 / \partial p_1 = \partial P_1 + P_2 + P_3 / \partial p_2 = 0$ (p_3 fixed) so $\partial P_1 + P_2 / \partial p_1 \leq 0$, since $\partial P_3 / \partial p_1 \geq 0$. The point is on the right of the point where $\partial P_1 + P_2 / \partial p_1 = 0$ (on C_2).

Lemma.

When p_3 increases the unique point on the bisector, where $\partial P_3 / \partial p_3 = 0$, moves to the right.

Oner writes: $\partial P_3 (p, p, p_3) = 0$. One differentiates: $2 \partial^2 P_3 /\partial p_3 \partial p_1 dp + \partial^2 P_3 / \partial p_3^2 dp_3 = 0$. The second term is negative (concavity of P_3 (p_1, p_2, p_3) for $p_1 = p_2 = p$, p_3 varying, and $dp_3 \geq 0$). Therefore, $dp \geq 0$ if $\partial^2 P_3 /\partial p_3 \partial p_1 \geq 0$. For $p_2 = p$ fixed, consider the plane O p_1 p_3. The point where $\partial P_3 / \partial p_3 = 0$ is on the reaction function R_3, with a positive slope. Therefore

$\partial^2 P_3 /\partial p_3 \partial p_1 \geq O$ (one differentiates the equation of R_3 $\partial P_3 / \partial p_3 = 0$ in the plane O p_1 p_3).

A point such that $\partial P_3 / \partial p_3 = 0$ on the bisector (p_3 fixed), exists, because for any $p_1 = p_2$ fixed, when p_3 increases P_3 varies from 0 ($p_3 = 0$) to 0 ($p_3 = 1$). And this point is unique. The intersection of the surface $\partial P_3 / \partial p_3 = 0$ and the plane $u_1 = u_2$ is an increasing curve.

When p_3 starts increasing from 0, the point $\partial P_3 / \partial p_3 = 0$ meets C_1, first, in M_1, the Nash equilibrium with three products: $\partial P_1 / \partial p_1 = \partial P_2 / \partial p_2 = \partial P_3 / \partial p_3 = 0$. Then, p_3 increasing, it meets C_2 in P_2, representing the equilibrium of E selling the products 1 and 2, and E_3 selling the product 3: $\partial P_1 + P_2 / \partial p_1 = \partial P_1 + P_2 / \partial p_2 = 0$ and $\partial P_3 / \partial p_3 = 0$. Then, p_3 increasing, the point is on the right of Q_3, the choice of the monopoly selling three products, on the bisector $p_1 = p_2 = p_3$. Indeed, in Q_3, $\partial P_1 + P_2 + P_3 / \partial p_3 = 0$, therefore $\partial P_3 / \partial p_3 \leq 0$ since $\partial P_1 + P_2 / \partial p_3 \geq 0$.

Proof that the slope of C_1 is positive.

One differentiates $\partial P_1 / \partial p_1 = 0$ in p_1 and p_3: $\partial^2 P_1 / \partial p_1^2 dp_1 + \partial^2 P_1 / \partial p_1 \partial p_3 dp_3 = 0$. For $dp_3 \geq 0$, $dp_1 \geq 0$ because: $\partial^2 P_1 / \partial p_1^2 \leq 0$ (concavity of P_1 (p_1, p_3) in p_1 for p_2 fixed), and $\partial^2 P_1 / \partial p_1 \partial p_3 \geq 0$ (this is demonstrated considering the plane p_2 fixed, the axes O p_1 p_3, the point is on the reaction function R_1which has a positive slope).The curve of the reaction function R_1 is moving to the right, the curve R_2 is moving upward, the equilibrium point is moving on the right on the bisector $p_1 = p_2$.

The slope of C_1 is infinite at the point R_1. Indeed, $\partial^2 P_1 / \partial p_1 \partial p_3$ is equal to 0 at this point.

$\partial^2 P_1 / \partial p_1 \partial p_3 = \partial D_1 / \partial p_3 + p_1 \partial^2 D_1 / \partial p_1 \partial p_3$. Also, $\partial D_1 / \partial p_3 = \partial D_3 / \partial p_1$. As D_3 (p_1, p_2, 1) = 0 for any p_1 and p_2, $\partial D_3 / \partial p_1 = 0$, $\partial D_1 / \partial p_3 = 0$ and $\partial^2 D_1 / \partial p_1 \partial p_3 = \partial^2 D_3 / \partial p_1^2 = 0$.

Proof that the slope of C_2 is positive.

Indeed, the sign of the slope of C_2 is not known. It depends on the sign of $\partial^2 P_1 + P_2 / \partial p_1 \partial p_3$ (p, p, p_3). The sign of the slope is the same. One is sure that the slope is positive in R_2. It has been already demonstrated that $\partial^2 P_1 / \partial p_1 \partial p_3 = 0$ if $p_3 = 1$. And $\partial^2 P_2 / \partial p_1 \partial p_3 = p_2 \partial^2 D_2 / \partial p_1 \partial p_3$ is positive. In general, $\partial^2 D_i / \partial p_j \partial p_k$ is positive. It is easily demonstrated. It is a consequence of the definition of the demands (the "pockets").

If one supposes the curve C_2 monotonic, it is increasing. It is to add a hypothesis. In any way the sign of the slope of C_2 does not matter very much, given the purpose of the article (what matters is the sign of the slope of C_3).

The sign of the slope of C_2 at P_2 is uncertain.

What is demonstrated in the article of the author [1] is this: when the close down is profitable, with obvious notation, $p_{independent}$ increases as far as a common value with p_{kept} while p_{closed} increases as far as 1. If the common value is reached with p_{kept} increasing, R_1 is on the right of P_2. It is obvious if P_2 is above the bisector ($p_3 \geq p_1$). But P_2 is below the bisector (as it is represented on the figure 5). At P_2, $\partial P_3 / \partial p_3 = 0$ and $\partial P_1 / \partial p_1 \leq 0$. In a plane p_2 fixed, in the axes O p_1 p_3, P_2 is on the reaction function R_3, with $\partial P_1 / \partial p_1 \leq 0$, it is on the strand NE of R_3 and $p_1 \geq p_3$. One does not know if R_1 is on the left or on the right of P_2. If it is on the right, the curve C_2 (supposed monotonic) is increasing, because R_2 is on the right of R_1. And the close down makes the consumers' surplus decrease (the points P_2 and R_1 do not describe the close down, but the consumers' surplus at P_2 and R_1 are the consumers' surpluses before and after the close down).

The point R_1 could be on the left of P_2.

If the point R_1 is on the left of P_2, given that the slope of C_2 is positive at R_2, the curve C_2 (supposed monotonic) is increasing.

Proof that the slope of C3 is negative.

The demonstration is awkward and is put in Appendix 1.

4. A Tractable, Particular Case

In this example, the density of utility is homogeneous, equal to 1.

The formulas giving the demands depend on the values of p_1, p_2, p_3. There are regions in a plane p_3 fixed (see figure 6). For instance, a region is: $p_2 \leq p_1 \leq p_3$.

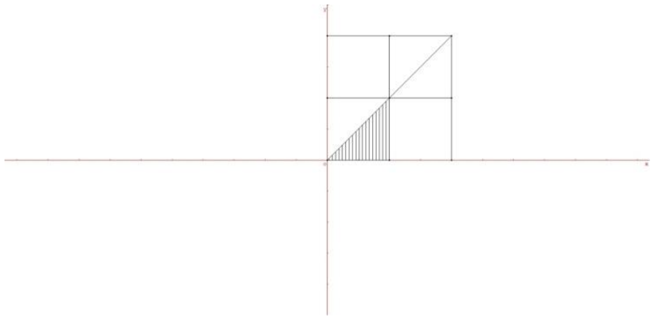

Figure 6. The six regions in the plane O p_1 p_2 with $0 \leq p_i \leq 1$ when $p_3 = 1 / 2$. The hatched region corresponds to $p_2 \leq p_1 \leq p_3$

In these conditions, the formulas of the demands are:

$D_1 = p_3^3 / 6 - p_2 p_3^2 / 2 + p_1^2 / 2 - p_3^2 / 2 - p_1 p_2 + p_2 p_3 - p_1 + p_2 / 2 + p_3 / 2 + 1 / 3$.

$D_2 = p_3^3 / 6 - p_1 p_3^2 / 2 - p_3^2 / 2 - p_1^2 / 2 + p_1 p_3 + p_3 / 2 - p_2 + p_1 / 2 + 1 / 3$.

$D_3 = (1 - p_3) [1 / 3 + p_1 + p_2 / 2 + p_1 p_2 - (2 / 3 + p_1 + p_2 / 2) p_3 + p_3^3 / 3]$.

To have the formulas in the other regions, one switches the indexes.

Now one finds the four equilibrium points N (Nash equilibrium with three products), N' (Nash equilibrium with two products), P (monopoly selling three products), and P' (monopoly selling two products).

Nash equilibrium with three products (N).

One uses the formulas above. One finds $p_0 \approx 0{,}32$. The joint profit is $P_N \approx 0{,}31$.

Monopoly selling three products (P).

One finds $p_0 \approx 0{,}63$. The joint profit is $P_P \approx 0{,}47$.

If $p_3 = 1$, one considers in axes O u_1 u_2 the areal density equal to 1 (figure 7). The calculations have been already presented by the author [4]. One can find the demands D_1 and D_2, making $p_3 = 1$ in the formulas above.

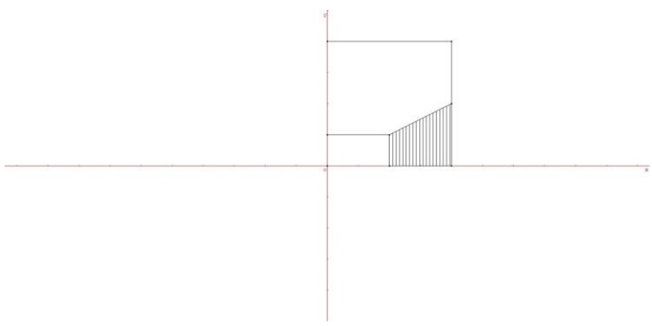

Figure 7. The density is areal and equal to 1. The demand D_1 is shown for $p_1 = 1 /2$ and $p_2 = 1 / 4$

One finds $p_{N'} \approx 0{,}41$ and the joint profit is $P_{N'} \approx 0{,}36$.

Monopoly selling two products (P').

One finds $p_0 \approx 0{,}57$ and the joint profit is $P_{P'} \approx 0{,}47$.

The four points are in the order described in the model. One has from the left to the right: N (0,32, 0,32, 0,32), N' (0,36, 0,36, 1), P' (0,57, 0,57, 1), P (0,63, 0,63, 0,63).

The equilibrium is unstable. The joint profit for the Nash equilibrium with three products (0,31) is less than the joint profit with two products (0,36). But the buy and close down is not profitable: 2 / 3 of P_N, (0,20), is more than 1 / 2 of $P_{N'}$, (0,18). Therefore, all what can happen is the withdrawal of a product thanks to lateral payments. But it is difficult [1]. It remains that the regulator can examine the question of the second best.

What is the second best?

It is the Nash equilibrium with two products.

As the demands derive from a potential, any path can be used to calculate Δ CS, the variation of the consumers' surplus between two points. One chooses the path: N' → (1, 1, 1) → P.

One has to calculate:

$\Delta CS = - \int (1 - p^2)\, dp_{\, N' \to (1, 1, 1)} + \int (1 - p^3)\, dp_{\, (1, 1, 1) \to P}$.

$\Delta CS = [p - p^3 / 3]^1_{0,41} + [p - p^4 / 4]^1_{0,63}$.

$\Delta CS \approx - 0{,}12$.

Another interesting question is: does the monopoly selling two products create consumers' surplus when it sells a product more?

The answer is yes.

One has to calculate:

$\Delta CS = - \int (1 - p^2)\, dp_{\, P' \to (1, 1, 1)} + \int (1 - p^3)\, dp_{\, (1, 1, 1) \to P}$

$\Delta CS = [p - p^3 / 3]^1_{0,57} + [p - p^4 / 4]^1_{0,63}$

The quantity Δ CS is very small but positive: $\Delta CS \approx 0{,}006$.

So, the studied example corresponds to the intermediate case described above. The second best is the Nash equilibrium with two products. The monopoly selling two products creates consumers' surplus when it sells a new product. In other words: the diversity of the products sold is real, but it is not enough to have the monopoly being the second best.

One has also to demonstrate that the tractable example has the characteristics of the model: the prices are strategic complements. As the demonstration is a little awkward, it has been put in the Appendix 2.

5. Conclusions

Interaction is not a catchword. It is a complex phenomenon, which deserves an accurate modelling. In this article the chosen model is Bertrand competition, the demands being deduced from the consumers' utilities. One has made several hypotheses: (1) the reaction functions exist and are increasing (2) symmetry of the consumers' utilities about the three planes $u_i = u_j$ (3) unicity of the equilibria. In these conditions, it is difficult to state that one affords an accurate twin of the reality. Rather, an "experience of thought" is proposed. That game theory allows "experiences of thought" is set out by Ariel Rubinstein, a specialist of game theory [5]. A game theory model makes ponder: effects are isolated, studied and the model says how they add to give a kind of result. But it is an artefact, useful to allow reasonings.

An example is the "experience of thought" proposed in this article.

A multiproduct monopoly has a strength and a weakness:

- The strength is that it keeps the diversity of the products sold. A monopoly sells a new product as soon as it can, since it makes more profit. And it never closes down an

asset (cancelling the sale of a kind of product). It is easily explained. One compares a multiproduct monopoly and a multiproduct firm with a competitor. The multiproduct firm with a competitor can possibly close down an asset, making a profit, because … it has a competitor. When the asset is closed down, the competitor increases its price (it is the strategic effect) and then, the firm which sells only one product, now, makes a gain. And it chooses a new price to optimize its profit. The strategic effect compensates and besides the loss due to the closing down (direct effect). It cannot occur if a multiproduct monopoly is concerned.
- The weakness is high prices. The multiproduct monopoly does not fear the stealing of market shares: in case of stealing of a market share, there is a gain compensating the loss, since the product the market share of which increases, is also owned by the multiproduct monopoly.

Finally, as one or the other of the two opposite effects is stronger, one has the three situations already described:

- The second best is the monopoly (selling three products). The diversity of the products sold matters. Also, if the multiproduct monopoly selling two products sells a new product, the consumers' surplus increases. In some way, this is a criterium for a real diversity (differentiation of the products sold): when the monopoly sells a new product, selling three products, the consumers' surplus increases.
- There is an intermediate situation. The diversity of the products sold is real (when the monopoly sells a new product, the consumers" surplus increases), but it is not enough. The prices are high. The Nash equilibrium (with two products) creates more consumers' surplus.
- The second best is the Nash equilibrium (with two products). There is not a real diversity of the products sold. When the monopoly sells a new product, it does not create more consumers' surplus.

So, two interesting criteria have appeared:

- A criterium for the Nash equilibrium (with three products) stable. It is: the entry of the third product increases the joint profit [1].
- A criterium for the diversity (differentiation of the products sold). It is: when the multiproduct monopoly sells a new product, the consumers' surplus increases.

Notice that when the second best is the monopoly (selling three products) the close down triggers a decrease of the consumers 'surplus. It is easily demonstrated, considering the consumers' surplus during a chain of operations: $P_2 \rightarrow R_1 \rightarrow Q_3 \rightarrow P_2$. This has a meaning for the regulator. Suppose he is coping with a product possibly withdrawn from the market (because the close down is profitable). If he must keep the diversity of the products sold (because the second best is the monopoly selling three products), there is a sign for him: the close down makes decrease the consumers' surplus.

Of course, empirical studies are needed to check the validity of these criteria. For instance, such a study is the article quoted in the author's article [1]. It is "Antitrust and innovation: welcoming and protecting disruption" [6]. So, the experience of thought allows reasonings and explanations, and also provides questions. Empirical studies complete the experience of thought, if they deal with the questions which are arisen by the experience of thought. In particular, it would be interesting to empirically study this question: does a multiproduct monopoly selling a new product, creates more consumers' surplus?

We live in a "liquid society" [7]. All our choices are transitory and liquid: the city where we live, the job, the consumers' s tastes etc. The assets are also liquid. Not only in the financial sense: an asset is liquid when it can be sold immediately, for a sum of money. But in this sense: an asset is deliberately close down, without any reason of bankruptcy. The author has already commented some cases [8]. The management of multiproduct firms can be very complex. In a portfolio of brands, there can be … thousands of brands [9]. Suppose a multiproduct firm closing down an asset. It is because it drives the prices down. The competitors outside the multiproduct firm, will increase their prices. It remains a competitor, inside the multiproduct firm, after the closing down. It benefits from the increase of prices (of the competitors outside the multiproduct firm). And the loss (due to the close down) is compensated and besides, by the gain.

There are two possible reasons when an asset can be closed down, and it is profitable: (1) the product generates a small utility, so the seller chooses low prices, to succeed in sales (2) the product is weakly differentiated from another product sold. This triggers a price war.

Appendix 1

Proof that the slope of the curve C_3 is negative.

One starts demonstrating that if $p_3 = 1$, $\partial^2 P_1 + P_2 + P_3 / \partial p_1 \partial p_3$ is negative.

The quantity $\partial^2 P_1 + P_2 + P_3 / \partial p_1 \partial p_3$ can be written (at the point (p, p, 1):

$\partial D_1 / \partial p_3 + \partial D_3 / \partial p_1 + p\,[\partial^2 D_1 + D_2 + D_3 / \partial p_1 \partial p_3] + (1 - p)\,\partial^2 D_3 / \partial p_1 \partial p_3$.

$\partial D_1 / \partial p_3 = \partial D_3 / \partial p_1$ is equal to 0 ($D_3 = 0$ if $p_3 = 1$).

The quantity between brackets is negative. It is a consequence of the definition of the demands (the "pockets").

And $\partial^2 D_3 / \partial p_1 \partial p_3$ is negative if $p_3 = 1$.

If $p_3 = 1$, $\partial D_3 / \partial p_1\, d p_1 = 0$ (with $d p_1 \geq 0$).

If p_3 is just a little below 1 ($d p_3 \leq 0$):

$\partial D_3 / \partial p_1\, d p_1 > 0$.

Therefore, $\partial^2 D_3 / \partial p_1 \partial p_3$ is negative.

Then one demonstrates that in Q_3, $\partial^2 P_1 + P_2 + P_3 / \partial p_1 \partial p_3 \leq O$.

In Q_3 there is the unique maximum of $P_1 + P_2 + P_3$. Given the symmetry one has:

$\partial^2 P_1 + P_2 + P_3 / \partial p_i^2 = a$ and $\partial^2 P_1 + P_2 + P_3 / \partial p_i \partial p_j = b$.

The Hessian matrix has for coefficients $a_{ii} = a$ and $a_{ij} = b$. An obvious eigen value is $b - a$.

Therefore $b - a \leq 0$ and a being negative, $b \leq 0$.

The slope of C_3 in Q_3 and R_2 is negative.

One has to make one of these hypotheses:

- C_3 is monotonic
- C_3 is on one side, left of right, of the vertical passing by Q_3.
- Considering the vertical passing by Q_3, the value of $\partial^2 P_1 + P_2 + P_3 / \partial p_1 \partial p_3$ is negative when $p_3 = 1$ and in Q_3. Suppose that this value is negative at any point of the segment. R_2 cannot be on the right side of the vertical. C_3 should cross the vertical, starting to the left from Q_3, then reaching the point R_2. And at the point of intersection, the slope should be positive. But this is impossible.

Accepting one of these hypotheses, the consequence is that the strand of C_3 is on the left side of the vertical. And R_2 is on the left of Q_3.

Appendix 2

One wants to prove that the curve R_1 (p_3 fixed), in the plane O p_1 p_2, is increasing. One studies the SE strand. At the start, when $p_3 = 1$, it is easy to prove it. The prices are strategic complements (figure 8).

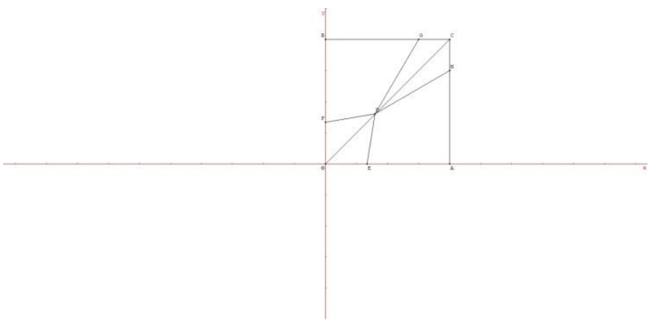

Figure 8. The reaction functions R_1 and R_2 when $p_3 = 1$ are shown. The shape is not linear. The strand SE, which is studied, goes from (1/3, 0) to (0,41, 0,41)

There is a discontinuity of the slope on the bisector, at the point which is the Nash equilibrium (0,41, 0,41). One has to prove that when p_3 decreases, the curves R_1 (p_3) remain increasing.

The strand SW is symmetrical (about the bisector). One needs also to study strand NE, the strand NW being symmetrical (about the bisector).

To study strand SE, one uses the formulas already set out (the case $p_2 \leq p_1 \leq p_3$).

The demonstration is awkward and needs several steps.

Step 1.

Lemma. The curve R_1 (p_3) cuts the bisector at a unique point (p_0, p_0), p_0 decreasing when p_3 decreases.

The formula for R_1 (p_3) is:

$\partial P_1 / \partial p_1 (p_1, p_2, p_3) = p_3^3 / 6 - p_2 p_3^2 / 2 + 3 / 2 p_1^2 - p_3^2 / 2 - 2 p_1 p_2 + p_2 p_3 - 2 p_1 + p_2 / 2 + p_3 / 2 + 1 / 3$.

Or:

$\partial P_1 / \partial p_1 (p_1, p_2, p_3) = A (p_1, p_3) + p_2 B (p_1, p_3)$

$A (p_1, p_3) = p_3^3 / 6 + 3 / 2 p_1^2 - p_3^2 / 2 - 2 p_1 + p_3 / 2 + 1 / 3$.

$B (p_1, p_3) = - p_3^2 / 2 - 2 p_1 + p_3 + 1 / 2$.

To find the intersection point of R_1 and the bisector one writes: $p_1 = p_2 = p_0$.

$\partial P_1 / \partial p_1 (p_0, p_0, p_3) = p_3^3 / 6 + 3 / 2 p_0^2 - p_3^2 / 2 - 2 p_0 + p_3 / 2 + 1 / 3 + p_0 (- p_3^2 / 2 - 2 p_0 + p_3 + 1 / 2)$.

One demonstrates that these is a unique root p_0, $0 \leq p_0 \leq 1$, for p_3 fixed.

Also:

$\partial^2 P_1 / \partial p_1 \partial p_0 = - p_0 - 1 / 2 (p_3 - 1)^2 - 1$, negative

$\partial^2 P_1 / \partial p_1 \partial p_3 = 1 / 2 (p_3 - 1)^2 + p_0 (1 - p_3)$, positive.

Therefore, when $dp_3 \leq 0$, $d p_0 \leq 0$. When p_3 decreases from 1, the intersection point moves towards the left. At some time p_3 "catches up" with p_0, $p_3 = p_0 = 0{,}32$. It corresponds to the Nash equilibrium with three products. The used formulas ($p_2 \leq p_1 \leq p3$) are no longer valid. But it does not matter because the equilibrium points which are calculated correspond to $p_3 > 0{,}32$.

Step 2.

Lemma. When p_3 decreases from 1 to 0,32, B remains positive.

B decreases from 0,18 (when $p_3 = 1$) to 0,13 (when $p_3 = 0{,}32$).

To prove that B decreases:

$d B = - 2 d p_0 + (1 - p_3) d p_3$

$\partial^2 P_1 / \partial p_1 \partial p_0 \, d p_0 + \partial^2 P_1 / \partial p_1 \partial p_3 \, d p_3 = 0$.

d B is of the same sign than a quantity which is negative.

Therefore, $B = \partial^2 P_1 / \partial p_1 \partial p_2$ is positive.

And: $\partial^2 P_1 / \partial p_1^2 = 3 p_1 - 2 - 2 p_2$, $\partial^2 P_1 / \partial p_1^2 (p_0, p_0, p_3) = p_0 - 2 \leq 0$.

The slope of R_1 (on the bisector) $- \partial^2 P_1 / \partial p_1^2 / \partial^2 P_1 / \partial p_1 \partial p_2$ is positive.

The slope of R_1 at the intersection with the bisector, is positive.

The curve R_1 passes below the bisector, since R_1 reaches $p_2 = 0$ without cutting the bisector, since there is a unique root of the equation $\partial P_1 / \partial p_1 (p_0, p_0, p_3) = 0$.

The strand of R_1 is entirely in the triangle ((0, 0), (p_0, p_0), (p_0, 0)). Indeed, $p_2 = - A (p_1, p_3) / B (p_1, p_3)$, for $p_1 = p_0$ shows that there is a unique point of R_1 on the vertical passing by (p_0, p_0). This point is (p_0, p_0).

Step 3.

The curve R_1 is monotonic.

It is impossible that:

$\partial P_1 / \partial p_1 (p_1, p_2, p_3) = 0$, and

$\partial P_1 / \partial p_1 (p'_1, p_2, p_3) = 0$, with p_1 and p'_1 different.

One subtracts. A necessary condition is:

$3 / 2 (p_1 + p'_1) - 2 - 2 p_2 = 0$.

This is impossible. Since R_1 is in the triangle ((0, 0), (p_0, p_0), (p_0, 0)), $p_1 \leq p_0$, $p_2 \leq p_0$, and $p_0 \leq 0{,}41$.

A majorant is $3 \times 0{,}41 - 2 - 2 p_2 < 0$.

This quantity is negative, not equal to 0.

The strand starts from $p_2 = 0$, is increasing as far as the point on the bisector (p_0, p_0).

The prices are strategic complements.

The strand slides toward the left when p_3 decreases (from $p_3 = 1$ to $p_3 = 0{,}32$).

Indeed:

$\partial^2 P_1 / \partial p_1^2 = 3 p_1 - 2 p_2 - 2 < 0$ (one has a majorant which is negative, replacing p_1 by 0,41).

$\partial^2 P_1 / \partial p_1 \partial p_3 = 1 / 2 (p_3 - 1)^2 + p_2 (1 - p_3) > O$.

The strand slides starting from (1 / 3, 0) to (0,41, 0,41) when $p_3 = 1$, to (0,27, 0) to (0,32, 0,32) when $p_3 = 0{,}32$.

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