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Method of Estimating the Transition Matrix (M) of the Mover-Stayer Model When Rate of Transition Follows Negative Binomial Distribution

Yusuf J. Adams[1,*], Sauta S. Abdulkadir[2], Hamidu U. Waniyos[3]

[1]Statistics Office, CBN Yola, Nigeria
[2]Modibbo Adama University, Yola, Nigeria
[3]Adamawa State Polytechnic, Yola, Nigeria

Abstract Motivated by the work of Spilerman (1972) and that of Johnson, Kotz and Kemp (1992), on extension of the Mover-Stayer model, Adams and Abdulkadir (2018) assumed negative binomial distribution for π_i to model rate of transition in Poisson distribution, which gave the Polya-Aeppli distribution as a mixture. However, that paper did not specify a method or derive an expression for estimating the transition matrix M. This paper attempts to provide a method of estimating the transition matrix M, to compliment the work earlier done by Adams and Abdulkadir (2018). In addition, an attempt was made to obtain an estimate of the stayer population. The obtained expressions were tested using simulated data adopted from Spilerman (1972).

Keywords Mover-Stayer, Transition matrix, Polya-Aeppli, Stochastic, Negative Binomial

1. Introduction

The extension of the Mover-Stayer Model proposed by Blumen, Kogan and McCarthy (1955) is still an active area of research. Spilerman (1972) extended the basic model by specifying gamma distribution for the transition rate, the mixture of which resulted in Negative Binomial distribution. Due to some observed shortcomings of the Negative Binomial distribution that may not capture situations where excess zeroes exist in the distribution of movements, Adams and Abdulkadir (2018) extended the model by using Negative Binomial distribution to model rate of transition in Poisson distribution, which gave the Polya-Aeppli distribution as a mixture.

Motivated by the work of Spilerman (1972) and Johnson, Kotz and Kemp (1992), they assume negative binomial for π_i. The choice of the negative binomial as a mixing distribution was informed by considering the number of transitions required to achieved desire events

$$\pi_i = f(\lambda) = \binom{k-1}{v-1}(1-\rho)^v \rho^{k-v}. \tag{1}$$

k = 1, 2, …..

$$r_v(t) = \sum_{v-1}^{\infty} \frac{(\lambda t)^v e^{-\lambda t}}{v!}\binom{k-1}{v-1}(1-\rho)^v \rho^{k-v} \tag{2}$$

$$r_v(t) = e^{-\lambda t}\sum_{v=1}^{\infty} \frac{\{\lambda t(1-\rho)\}^v}{v!}\binom{k-1}{v-1}\rho^{k-v}; \tag{3}$$

k =1, 2, ……….

The equation (3) coincides with Polya-Aeppli distribution, where the sum stops for $v > k$. If the parameter $\rho = 0$, the distribution in (3) reduces to the classical homogenous Poisson distribution (Minkova, 2002; Minkova, 2004; Chukova and Minkova, 2012).

Consider individuals with common rate of movement equal to λ at time t, the one-step probability matrix would be given by

$$P(1) = \sum_{v=0}^{\infty} r_v(1) M^v \tag{4}$$

Where $r_v(1)$ is Poisson distribution with t = 1. The transition matrix is obtained by simplifying for M in (4) above.

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2n} \\ P_{31} & P_{32} & P_{33} & \cdots & P_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ P_{n1} & P_{n2} & P_{n3} & \cdots & P_{nn} \end{bmatrix} \tag{5}$$

* Corresponding author:
adamyusjib@gmail.com (Yusuf J. Adams)
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P is a stochastic matrix, hence

$$\sum_{j=1}^{n} P_{1j} = \sum_{j=1}^{n} P_{2j} = \dots\dots = \sum_{j=1}^{n} P_{nj} = 1 \qquad (6)$$

However, the work done by Adams and Abdulkadir (2018) did not specify a method estimating the transition matrix M or derive an expression for the transition matrix M. Consequently, this paper set out to obtain that expression as well as estimate the stayer population using simulated data.

2. Methodology

2.1. Obtaining an Expression for the Transition Matrix M

Under the assumption that each individual's transitions follow a Poisson process, with the individual rates of mobility specified by a negative binomial density, the proportion of the population making v moves in (0,t) will satisfy a Polya-Aeppli distribution.

To obtain an expression for M, the transition matrix for the mover individuals, we used equation (3) in (4)

$$P(1) = \sum_{v=0}^{\infty} e^{-\lambda} \sum_{v=1}^{\infty} \frac{\{\lambda(1-\rho)\}^{v}}{v!} \binom{k-1}{v-1} \rho^{k-v} M^{v} \quad (7)$$

$$P(1) = e^{-\lambda} \sum_{v=0}^{\infty} \sum_{v=1}^{\infty} \frac{\{\lambda(1-\rho)\}^{v}}{v!} \binom{k-1}{v-1} \rho^{k-v} M^{v} \quad (8)$$

Using convergence of infinite sum,

$$\sum_{v=0}^{\infty} \frac{\{\lambda(1-\rho)\}^{v}}{v!} = e^{\lambda(1-\rho)} \qquad (9)$$

$$P(1) = e^{-\lambda} e^{\lambda(1-\rho)} \sum_{v=1}^{\infty} \binom{k-1}{v-1} \rho^{k-v} M^{v} \quad (10)$$

$$P(1) = e^{-\lambda+\lambda-\lambda\rho} \sum_{v=1}^{\infty} \binom{k-1}{v-1} \rho^{k-v} M^{v} \quad (11)$$

$$= e^{-\lambda\rho} \sum_{v=1}^{\infty} \binom{k-1}{v-1} \rho^{k-v} M^{v} \qquad (12)$$

Let $k - v = (k-1) - (v-1) = \alpha$

$$P(1) = e^{-\lambda\rho} \sum_{v=1}^{\infty} \binom{\alpha+v-1}{v-1} \rho^{(k-1)-(v-1)} M^{v}$$

$$P(1) = e^{-\lambda\rho} \sum_{\alpha=0}^{\infty} \binom{\alpha+v-1}{\alpha} \rho^{\alpha} M^{v} \qquad (13)$$

Since $\binom{n}{r} = \binom{n}{n-r}$

$$P(1) = e^{-\lambda\rho} M^{v} \sum_{\alpha=0}^{\infty} \binom{\alpha+v-1}{\alpha} \rho^{\alpha} \qquad (14)$$

For ρ, a scalar, and any real number v; the following binomial expansion holds

For ρ, a scalar, and any real number v; the following binomial expansion holds

$$\sum_{\alpha=0}^{\infty} \binom{\alpha+v-1}{\alpha} \rho^{\alpha} = (1-\rho)^{-v} \qquad (15)$$

where the condition for convergence of the infinite sum in (15) is $|\rho| < 1$.

$$P(1) = e^{-\lambda\rho} M^{v} (1-\rho)^{-v} \qquad (16)$$

For $v = 1$, equation (16) becomes

$$P(1) = e^{-\lambda\rho} M (1-\rho)^{-1} \qquad (17)$$

$$\therefore \; M = (1-\rho) e^{\lambda\rho} P(1) \qquad (18)$$

Equation (18) therefore provides a method for estimating M from the population-transition matrix P(1), under the assumption that population heterogeneity in the rate of movement can be specified by a negative binomial density.

Furthermore, M is a stochastic matrix, as such,

$$\sum_{j=1}^{n} M_{1j} = \sum_{j=1}^{n} M_{2j} = \dots\dots\dots = \sum_{j=1}^{n} M_{nj} = 1$$

3. Estimation of Stayer Population

Blumen, Kogan, and McCarthy (1955) suggested decomposition of the matrix into two subpopulations: the movers and the stayers, because some persons are less apt to move than others in each time interval. Hence, the transition matrix for the entire population, P, is defined as

$$P(1) = S + (I - S)M \qquad (19)$$

According to Goodman (1961) and Morgan *et al* (1983), S has s_i (the proportion initially in the i^{th} state who are stayers) down the diagonal, and I is the identity matrix. Consequently,

$$P_{ii} = S_i + (1 - S_i) M_{ii} \qquad (20)$$

$$\Rightarrow M_{ii} = \frac{P_{ii} - S_i}{1 - S_i}; \; i = 1, 2, .., n \quad (21)$$

Using the estimator of m_{ii} presented above, we obtain the following estimator of s_i;

$$\therefore S_i = \frac{P_{ii} - M_{ii}}{1 - M_{ii}}; \text{ for } m_{ii} < 1 \qquad (22)$$

4. Testing the Estimated Equation Using Simulated Data

Table 4.1. Structure of The Simulated Data

A. INDIVIDUAL LEVEL TRANSITION MATRIX	B. DISTRIBUTION OF THE POPULATION BY RATE OF MOBILITY λ	Proportion of the Population with this λ value
$M = \begin{bmatrix} 0.600 & 0.200 & 0.100 & 0.100 \\ 0.150 & 0.700 & 0.100 & 0.050 \\ 0.100 & 0.100 & 0.750 & 0.050 \\ 0.050 & 0.050 & 0.100 & 0.800 \end{bmatrix}$	0.1	0.25
	1.0	0.35
	2.0	0.20
	3.0	0.10
	4.0	0.06
	5.0	0.04
		1.00

Source: Spilerman (1972)

4.1. Estimating the Expected Transition Matrix M

The "observed" data in Table 4.1, were then used with the equation

$$P(1) = \sum_{v=0}^{10} r_v(1) M^v \tag{23}$$

to generate an "observed" transition matrix P(1).

$$\Rightarrow P(1) = \begin{bmatrix} 0.650 & 0.156 & 0.101 & 0.093 \\ 0.118 & 0.719 & 0.101 & 0.063 \\ 0.090 & 0.102 & 0.747 & 0.061 \\ 0.056 & 0.064 & 0.101 & 0.779 \end{bmatrix} \tag{24}$$

The parameters of the model λ and ρ of the Polya Aeppli distribution can be estimated directly from observed data (moment estimates) on the number of moves by an individual. If $\bar{v}$ and σ_v^2 are the sampling mean and variance of this variable (number of moves), then estimates of λ and ρ can be obtained in terms of these values (see also Minkova 2012, p. 49). This yield

$$\hat{\rho} = \frac{\sigma^2 - \bar{v}}{\sigma^2 + \bar{v}} \text{ and } \hat{\lambda} = \frac{2\bar{v}^2}{\sigma^2 + \bar{v}} \tag{25}$$

Where $\bar{v} = 1.502$ and $\sigma^2 = 3.139$

Spilerman (1972), equally estimated α and β, the parameters of the negative binomial distribution using; $\hat{\boldsymbol{\beta}} = \frac{\bar{v}}{S^2 - \bar{v}}$ and $\hat{\alpha} = \hat{\beta}\bar{v}$; computed from the observed data.

Therefore, the estimated λ = 0.9722 and ρ = 0.3527, together with the observed P(1) matrix from equation (23), allow the M* matrix to be derived using equation (18),

$$\therefore M^* = \begin{bmatrix} 0.593 & 0.142 & 0.092 & 0.085 \\ 0.108 & 0.656 & 0.092 & 0.057 \\ 0.082 & 0.093 & 0.681 & 0.056 \\ 0.051 & 0.058 & 0.092 & 0.710 \end{bmatrix} \tag{26}$$

$\sum_{j=1}^{4} M_{1j} = 0.912$; $\sum_{j=1}^{4} M_{2j} = 0.913$;

$\sum_{j=1}^{4} M_{3j} = 0.912$; $\sum_{j=1}^{4} M_{4j} = 0.911$

4.2. The Stayer Population (S_i)

$$S_i = \frac{P_{ii} - M_{ii}}{1 - M_{ii}}$$

$$\therefore S_1 = \frac{P_{11} - M_{11}}{1 - M_{11}} = \frac{0.650 - 0.593}{1 - 0.593} = 0.140$$

Similarly,

$S_2 =$ 0.183; $S_3 =$ 0.207; $S_4 =$ 0.238

Consequently, the proportions per distribution of number of moves from simulated data, computed from individual-level transition matrix as proposed by Goodman (1961) and Morgan *et al* (1983) follows;

$$S = \begin{bmatrix} 0.140 & 0 & 0 & 0 \\ 0 & 0.183 & 0 & 0 \\ 0 & 0 & 0.207 & 0 \\ 0 & 0 & 0 & 0.238 \end{bmatrix} \tag{27}$$

5. Discussion and Conclusions

Adams and Abdulkadir (2018) used the Poisson distribution model as a baseline model to generate individual level transition matrix and the observed frequency distribution of moves. The expected frequencies were obtained using Polya-Aeppli distribution. Consequently, an expression was derived in this work, which provides a method for estimating M from the population-transition matrix P(1), under the assumption that population heterogeneity in the rate of movement can be specified by a negative binomial density. Hence, the proportion of stayer population per individual-level transition was computed.

The result shows that, 14.0% of the first baseline group retained that position throughout the period of study; 18.3% of the second baseline group retained that position throughout the period of study. Similarly, 20.7% and 23.8% of the third and fourth baseline groups retained that position throughout the period of study.

The results obtained were like that of Alawadhi and Konsowa (2010) who presented an application of Markov chain analysis of students flow at Kuwait University. They constructed a frequency matrix for the university, from which the transition probabilities were estimated. The matrix which represents the transition probabilities of remaining in or progressing to another state was also presented. Their paper however, did not estimate the stayer population.

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