International Journal of Astronomy 2021, 10(2): 50-59
DOI: 10.5923/j.astronomy.20211002.03

Rotating Disk Cosmological Systems in the Lagrangian and Relativistic Solutions

Adrián G. Cornejo

Electronics and Communications Engineering from Universidad Iberoamericana, Santa Rosa 719, CP 76138, Querétaro, Mexico

Abstract This work is a review that summarizes the previous works that describe the theory in which the movement of some circular or disk cosmological systems, such as the entire disk of the Solar System and spiral galaxies, corresponds to the dynamics like a rigid body in rotational motion. It has been found that both the Lagrangian solution and the relativistic solution consider all the energies and forces involved in the rotating systems. On this assumption, we have been derived the respective equations to explain the apsidal precession of the planets, the Earth's axial precession and the rotation curves of some spiral galaxies, all of them due to the effect of rotating systems like a rigid body in rotational motion. A good approximation between the calculated and the observations for each of these cases has been found. Thus, this theory unifies classical mechanics (through Lagrangian mechanics) with the solution of general relativity and its intrinsic consideration of the rigid body in rotational motion in the three-dimensional framework. Also, it unifies all these rotational behavior of these disk cosmological systems in the same concept of the rigid body in rotational motion.

Keywords Solar system: General, Planet-disk interactions, Galaxies: spiral, Kinematics and dynamics, Lagrangian mechanics, General Theory of Relativity

1. Introduction

Newton's law of gravitation [1,2] has been applied for hundreds of years with some success to explain several observed mechanical and astronomical phenomena, and it is still used to try to explain some problems in cosmology, although having some difficulties to provide an exact solution and confirm the measurements and observations in several cases.

The issue is that the classical total force only considers two terms, which are the Newtonian gravitational force and the centrifugal force [2]. Then, it has some limitations proper of this classical theory since it does not consider any other energy or force involved in the rotating systems. This is the reason why is necessary to add some matter or forces to adjust the calculations to the rotational velocities observed in spiral galaxies, such as the dark matter concept [3], which has not been detected to date, or even develop some adjustments like the proposed in the Modified Newtonian Dynamics (MOND) theory [4,5]. The main problem with these considerations is that the rotation of spiral galaxies and the entire Solar System (as circular or disk cosmological systems) have not been well taken into account, even in most of the standard astronomical considerations.

On the other hand, the net or absolute total force in the relativistic solution, which is derived from the General Theory of Relativity (GTR) [6], also considers a third additional term that includes the force related to the Coriolis force in a rotating system, which is inverse of the distance to the fourth power [7]. This is the main reason why the relativistic solution can be considered to provide an exact solution which does not require any additional matter and does not require any adjustment to fit to the rotational velocities observed in spiral galaxies.

This work summarizes the previous works that describe the theory in which the movement of some circular or disk cosmological systems, such as the entire disk of the Solar System and spiral galaxies, they all behave like a rigid body in rotational motion. Through those works, it has been found that both, Lagrangian and relativistic solutions consider all the forces and energies involved in the rotating systems. On this assumption, have been derived the respective equations to explain the apsidal precession of the planets [8], the Earth's axial precession [9,10] and the rotation curves of some spiral galaxies [11,12], if the gas is assumed to be a stable component of the galaxy, as described by the quasi-stationary density wave theory, which characterizes spirals as rigidly rotating, long-lived patterns (i.e. steady spirals) [13]. In the case of the barred spiral galaxies, the Kerr metric has been considered as part of the solution. Thus, the existence of barred spiral galaxies could be a confirmation of the Kerr metric. A good

* Corresponding author:
adriang.cornejo@gmail.com (Adrián G. Cornejo)
Received: Nov. 1, 2021; Accepted: Nov. 24, 2021; Published: Dec. 15, 2021
Published online at http://journal.sapub.org/astronomy

approximation between the calculated and the observations for each of these cases has been found. Thus, this theory unifies classical mechanics (through Lagrangian mechanics) with the solution of the general relativity in the three-dimensional framework. Also, it unifies all these rotational behavior of these disk cosmological systems in the same concept of the rigid body in rotational motion.

Although there may be several independent investigations to date regarding the existence of the supposed dark matter by indirect detection (that is, attributing some observations to dark matter), so far this is a proposed conjecture and its existence has not yet been confirmed or directly detected to date, so that its existence cannot yet be categorically confirmed. Therefore, the other theories that do not require dark matter to explain the rotation curves observed in spiral galaxies, such as the one considered here, should not be ruled out.

2. Lagrangian Solution for the Total Force of the Rotating System

2.1. Lagrangian Solution for the Rotating Frame of Reference

The rotating frame of reference is a frame that rotates around a spin axis (fixed for simplicity) with a given angular velocity. A rotating frame of reference is usually characterized by three inertial forces: the centrifugal force, the Coriolis force, and for a non-uniformly rotating frame of references, the Euler force [14]. The rotating frame of reference can be described through the general case of a rigid body in rotational motion and a fixed frame [15]. It is considered the distance from the spin axis to the final position of a point P that according to the fixed frame of reference is the radius vector named r_f (which will be considered as our generalized coordinate in the rotating system for the Lagrangian solution), while some position of the same point according to the rotating frame of reference is named r, and

$$r_f = r + R, \tag{1}$$

where R denotes the position from the origin of the rotating frame of reference according to the fixed frame. As usual, Lagrangian function summarizes the dynamics of the entire system, determining the kinetic and potential energies of the system, given by:

$$L = T - V, \tag{2}$$

where T is the total kinetic energy and V is the potential energy of the system [16]. Furthermore, the rotating frame of reference is described by the Lagrangian [17] for the motion of a particle with rest mass m in a rotating frame (with its origin coinciding with the fixed-frame origin, for simplicity) in the presence of the potential $U(r)$, given as:

$$L = \frac{1}{2} m \dot{r}_f^2 - U(r), \tag{3}$$

and the derivative of the generalized coordinate (1) is written as:

$$\dot{r}_f = \dot{r} + \dot{R} = \dot{r} + \Omega \times r, \tag{4}$$

where Ω is the angular velocity vector of the rotating frame of reference. Then, (3) can be written as:

$$L(r, \dot{r}) = \frac{1}{2} m \left| \dot{r} + \Omega \times r \right|^2 - U(r). \tag{5}$$

For the rotating system, the Euler-Lagrange equation is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_f}\right) - \frac{\partial L}{\partial r_f} = 0. \tag{6}$$

In addition, the equation for the canonical momentum in this system can be written as:

$$p = \frac{\partial L}{\partial \dot{r}_f} = m\left(\dot{r} + \Omega \times r\right), \tag{7}$$

and for the first part of (6), hence:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_f}\right) = m\left(\ddot{r} + \dot{\Omega} \times r + \Omega \times \dot{r}\right). \tag{8}$$

Moreover, for the second part of (6), the partial derivative is given by:

$$\frac{\partial L}{\partial r_f} = -\nabla U(r) - m\left[\Omega \times \dot{r} + \Omega \times \left(\Omega \times \dot{r}\right)\right]. \tag{9}$$

Thus, from (8) and (9), the Euler-Lagrange equation (6) for r_f term [18] is reduced as:

$$m\ddot{r} = -\nabla U(r) - m\left[\dot{\Omega} \times r + 2\Omega \times \dot{r} + \Omega \times \left(\Omega \times r\right)\right]. \tag{10}$$

In addition, the potential energy generates the fixed-frame acceleration, $-\nabla U = m a_f$, and from (10), for the acceleration can be written as:

$$a_r = a_f - A - \Omega \times \left(\Omega \times r\right) - 2\Omega \times v_r - \dot{\Omega} \times r, \tag{11}$$

which represents the sum of the net acceleration (a_f - A).

The centrifugal acceleration is given by $-\Omega \times (\Omega \times r)$, and $a_c = -2\Omega \times v_r$ is the Coriolis acceleration. When the final velocity v_f is equivalent to the Coriolis velocity v_c, yields:

$$v_c = v_f = 2\Omega \times r \therefore \Omega = \frac{v_f}{2r}. \tag{12}$$

In addition, from (11) the velocity is given as:

$$v_r^2 = v_f^2 - 2\Omega \times v_f r + \Omega^2 \times r^2 = \left(v_f - \Omega \times r\right)^2, \tag{13}$$

where simplifying, velocities in the two reference frames are given by the well-known equation:

$$v_f = v_r + \Omega \times r. \tag{14}$$

This means that the final velocity of a body rotating in a rotating frame of reference is the addition of both, the velocity of the body at the distance r from centre and the angular velocity of the rotating frame of reference at the same distance from the spin axis. The force in a body with rest mass m in the rotating system is given as:

$$F=\frac{mv_r^2}{r}=\frac{m}{r}\left(v_f-\Omega\times r\right)^2,$$
$$F=m\left(\frac{v_f^2}{r}-\Omega^2\times r+2\Omega\times v_f\right), \tag{15}$$

where the second term is the centrifugal force, and the third term $2m\Omega\times v_f$ is the Coriolis force.

2.2. Total Force in the Rotating Frame of Reference

As usual, the total energy in a rotating system is given by:

$$E=E_K+V(r), \tag{16}$$

where E_K is the kinetic energy and $V(r)$ is the effective gravitational potential energy. From (13) and (16), we can write:

$$E=\frac{1}{2}mv_r^2+V(r),$$
$$E=\frac{1}{2}m\left(v_f-\Omega\times r\right)^2+V(r). \tag{17}$$

And having the gravitational potential energy related to some central resting mass M and a given particle with rest mass m orbiting at a distance r, the total energy can be written as:

$$E=\frac{1}{2}m\left(v_f^2-2\Omega\times v_f r+\Omega^2\times r^2\right)-\frac{GMm}{r},$$
$$\frac{1}{2}mv_f^2=E-\left(-\frac{GMm}{r}+\frac{1}{2}m\Omega^2r^2-m\Omega\times v_f r\right), \tag{18}$$

where G is the Newtonian constant of gravity. Then, from (16), the effective gravitational potential energy is given as:

$$V(r)=-\frac{GMm}{r}+\frac{1}{2}m\Omega^2r^2-m\Omega\times v_f r. \tag{19}$$

Furthermore, the angular momentum of a test particle of rest mass m orbiting in a circular motion, in polar coordinates is defined as:

$$L=mrv=mr^2\omega, \tag{20}$$

where ω is the angular velocity of the body in a circular orbit. Also, considering the angular velocity of the rotating frame of reference, it can be written as:

$$L_\Omega=mr^2\Omega, \tag{21}$$

where Ω is the angular velocity of the rotating frame of reference. The second term of the right side of (19) corresponds to the angular velocity of the body orbiting around the centre, and the third term corresponds to the angular velocity of the body with respect to the rotating frame of reference. From this correspondence, (19) can be written as:

$$V(r)=-\frac{GMm}{r}+\frac{1}{2}m\omega^2r^2-m\Omega\times v_f r, \tag{22}$$

and multiply second and third terms, respectively, by terms that complement the parameters, which are equal to the unit (without changing its value or physical meaning), and replacing the angular velocity from (12), becomes:

$$V(r)=-\frac{GMm}{r}+\frac{1}{2}m\omega^2r^2\left(\frac{mr^2}{mr^2}\right)-m\Omega\times v_f r\left(\frac{m\omega\times v_f r}{m\omega\times v_f r}\right),$$
$$V(r)=-\frac{GMm}{r}+\frac{\left(mr^2\omega\right)^2}{2mr^2}-\left(\frac{v_f}{2r}\right)\frac{v_f^2m^2r^2\omega}{m\omega\times v_f r}. \tag{23}$$

Solving this equation, we can apply the equivalence between the escape velocity and the final orbital velocity [19] defined by:

$$v_e=\sqrt{\frac{2GM}{r}}=\sqrt{2}vf, \tag{24}$$

where v_e is the escape velocity. Also, considering the equivalence whit the orbital velocity v_0 [20] given by:

$$v_0=v_f=\sqrt{\frac{GM}{r}}, \tag{25}$$

and $v=\omega\times r$, and reducing common terms, (23) can be written as:

$$V(r)=-\frac{GMm}{r}+\frac{\left(mr^2\omega\right)^2}{2mr^2}-\frac{v_f^2r\left(mr^2\omega\right)^2}{mv_e^2r^3}. \tag{26}$$

From (20) and (25), we can write (26) in terms of the angular momentum of a body in a circular motion [21,22] as:

$$V(r)=-\frac{GMm}{r}+\frac{L^2}{2mr^2}-\frac{GML^2}{mv_e^2r^3}, \tag{27}$$

and by the radius derivative, the magnitude of the net or absolute total force is given by:

$$|F|=\frac{GMm}{r^2}-\frac{L^2}{mr^3}+\frac{3GML^2}{mv_e^2r^4}. \tag{28}$$

On the scenario where the escape velocity of a massive body (such as a black hole) is, as known, the speed of light c [8,23] and the escape radius is equivalent to the Schwarzschild radius [24], then we have the correspondence:

$$v_e^2=v_f^2\left(\frac{2r}{r_e}\right)\to c^2=v_f^2\left(\frac{2r}{r_S}\right), \tag{29}$$

where r_e is the escape radius and r_s is the Schwarzschild radius. Replacing (29) in (28), becomes:

$$|F|=\frac{GMm}{r^2}-\frac{L^2}{mr^3}+\frac{3GML^2}{mc^2r^4}. \tag{30}$$

The third term is related to the Coriolis force [7], which includes the inverse of the distance to the fourth power, which is derived from the GTR. This shows that the relativistic solution for the total force intrinsically considers the dynamics of a rotating disk system like a rigid body in rotational motion in the three-dimensional framework, which allows to unify classical mechanics (through Lagrangian mechanics) with the solution of the general relativity in the three-dimensional framework. Equations (28) and (30) are the same formula, and there is only one conversion factor, which is Eq. (29) referred to in [8,23], to express equation (28) as equation (30). These calculations are fully supported by Lagrangian mechanics, and it is easy to follow them by someone who knows this method.

3. Apsidal Precession in the Rotating Frame of Reference

Considering a body with rest mass m in a circular orbit at a distance r from the spin axis of a rotating system, precession results from the angular velocity of rotation and the angular velocity produced by the torque [8]. Then, the orbiting body will undergo precession expressed by the angular velocity of precession ω_φ as described for a rotating gyroscope of radius r [25], with some equivalencies defined as:

$$\omega_\varphi = \frac{d\varphi}{dt} = \frac{M_0}{L} = \frac{M_0}{I\omega} = \frac{mgr}{I\omega} = \frac{r \times F(r)}{mr^2\omega}, \tag{31}$$

where d_φ is the differential of the precession angle, dt is the time differential, ω is the angular velocity around the axis of rotation, I is the moment of inertia of a ring of radius r, M_0 is the modulus of angular momentum and g is the gravitational acceleration. When a rigid body rotates around its spin axis under an external force (for instance, given by $F(r) = m \cdot g$), the moment of the external forces is not null, and the angular momentum is non-conservative ($L \neq 0$). Then, the angular momentum changes direction with precession as an additional circular motion. In this way, considering the third term of (30) and the rotational velocity of a body in a circular motion, the angular momentum of inertia becomes:

$$M_0 = r \times F(r) = \frac{3GMmv\omega}{c^2}. \tag{32}$$

Substituting (32) into (31), hence:

$$\begin{aligned} \omega_\varphi &= \frac{d\varphi}{dt} = \left(\frac{3GMmv\omega}{c^2}\right)\frac{1}{mr^2\omega} = \left(\frac{3GMv}{c^2}\right)\frac{1}{r^2}, \\ d\varphi &= \frac{3GM\,\omega dt}{rc^2} = \frac{3GMd\theta}{rc^2}, \end{aligned} \tag{33}$$

where $d\theta$ is the differential of the radial angle covered by the rotation of the body in the rotating frame of reference. Extending (33) to the Kepler's geometry for a planet in an elliptical orbit around the Sun at a focus, where in the ellipse $\rho = a(1 - e^2)$, where a is the semi-major axis and e is the eccentricity. When the body moves in one revolution ($\theta = 2\pi$ radians), becomes:

$$d\varphi = \frac{3GM(2\pi)}{a\left(1-e^2\right)c^2} = \frac{6\pi GM}{a\left(1-e^2\right)c^2}. \tag{34}$$

Thus, considering the Newtonian equivalence of GM in terms of the period T for an elliptical orbit, defined as:

$$GM = \frac{(2\pi)^2 a^3}{T^2}, \tag{35}$$

and replacing in (34), becomes

$$d\varphi = \frac{3(2\pi)^3 a^2}{c^2T^2\left(1-e^2\right)} = \frac{24\pi^3 a^2}{c^2T^2\left(1-e^2\right)}, \tag{36}$$

which is the same relativistic equation to calculate the advance of Mercury's perihelion [26].

As known, by substituting the known data for Mercury [27] into (36), the advance of Mercury's perihelion gives about 43.013" arc per century (Figure 1). It is known that all planets precess, regardless of their distance from the Sun, and this equivalence can be applied to determine the apsidal precession of all other planets in the Solar System, giving the same results as the observations for each respective apsidal precession.

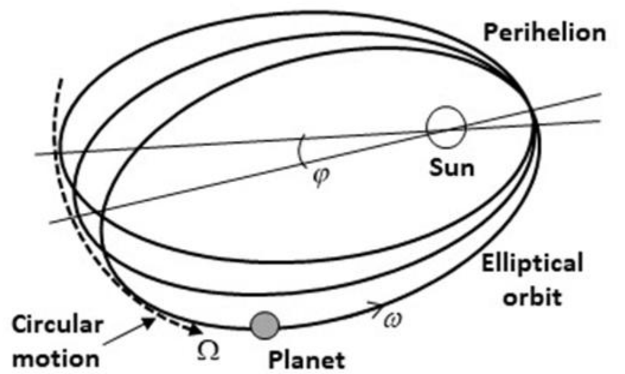

Figure 1. Precession of Mercury's perihelion in a rotating frame of reference with angular velocity Ω

4. Axial Precession in the Relativistic Solution

Considering that the entire disk of the Solar System could be rotating like a rigid body, the effect of that motion should be perceived from the Earth as a change in linear position with respect to the considered "fixed" stars, noting that these would have an apparent periodic circular motion [9,10]. From the third term of (30) for a rotating system, reducing common terms, becomes:

$$F_c = \frac{3GM\left(mr^2\omega_c\right)^2}{mc^2r^4} = \frac{3GMm\omega_c^2}{c^2}, \tag{37}$$

and for the acceleration, yields:

$$\mathrm{a}_c = \frac{3GM}{c^2}\omega_c^2, \tag{38}$$

where a_c is related to the Coriolis acceleration, which also has the following equivalences:

$$\mathrm{a}_c = 2\Omega \times \mathrm{a}t = 2\Omega \times v = 2\Omega^2 \times r. \tag{39}$$

Thus, integrating (39) two times with respect to time, hence:

$$x_c = \frac{1}{3}\Omega \mathrm{a}t^3 = \frac{1}{3}\Omega rt. \tag{40}$$

In the case of the free fall of a body, x_c is the displacement in the opposite direction to the Earth's rotation, and its velocity is given by:

$$v_c = \frac{x_c}{t} = \frac{1}{3}\Omega r. \tag{41}$$

The time to reach the ground is given by:

$$h = \frac{1}{2}gt^2 \therefore t = \left(\frac{2h}{g}\right)^{\frac{1}{2}}, \tag{42}$$

where h is the height and g is the gravitational acceleration. Substituting (42) into (40), the final displacement is written as:

$$x_c = \frac{1}{3}\Omega g\left(\frac{2h}{g}\right)^{\frac{3}{2}}. \tag{43}$$

To get an idea of the magnitude of this displacement x_c, let us consider an object at the equator free falling from a height of 100 meters. Substituting these values into (43) gives a total displacement of about 2.2 cm. This amount is small compared to the 100-meter drop, but it is certainly measurable. Furthermore, from (41) and with the equivalence $\Omega = v/r$ for a circular motion, we obtain:

$$\begin{aligned} \Omega r = v_c \rightarrow \Omega r &= \frac{1}{3}\omega_c r = \frac{1}{3}\left(\frac{v}{r}\right)r, \\ \omega_c &= \frac{v}{3r}. \end{aligned} \tag{44}$$

From Eqs. (38), (39) and (44), yields:

$$\begin{aligned} 2\Omega_v^2 \times r &= \frac{3GM}{c^2}\left(\frac{v}{3r}\right)^2, \\ \Omega_v &= \left(\frac{GMv^2}{6c^2r^3}\right)^{\frac{1}{2}}. \end{aligned} \tag{45}$$

Having the apsidal precession from (36), we must include this movement of Earth's orbit in the total angular frequency Ω_T of the system, giving:

$$\Omega_T = \Omega_v + \omega_\varphi = \left(\frac{GMv^2}{6c^2r^3}\right)^{\frac{1}{2}} + \frac{6\pi GM}{a\left(1-e^2\right)c^2t_\mathrm{P}}, \tag{46}$$

where t_P is the time in seconds of a sidereal year. For one revolution (2π radians), the total period is written as:

$$\begin{aligned} T_T &= \frac{2\pi}{\Omega_\mathrm{T}} = \frac{2\pi}{\Omega_v + \omega_\varphi}, \\ T_T &= \frac{2\pi}{\left(\frac{GMv^2}{6c^2r^3}\right)^{\frac{1}{2}} + \frac{6\pi GM}{a\left(1-e^2\right)c^2t_\mathrm{P}}}. \end{aligned} \tag{47}$$

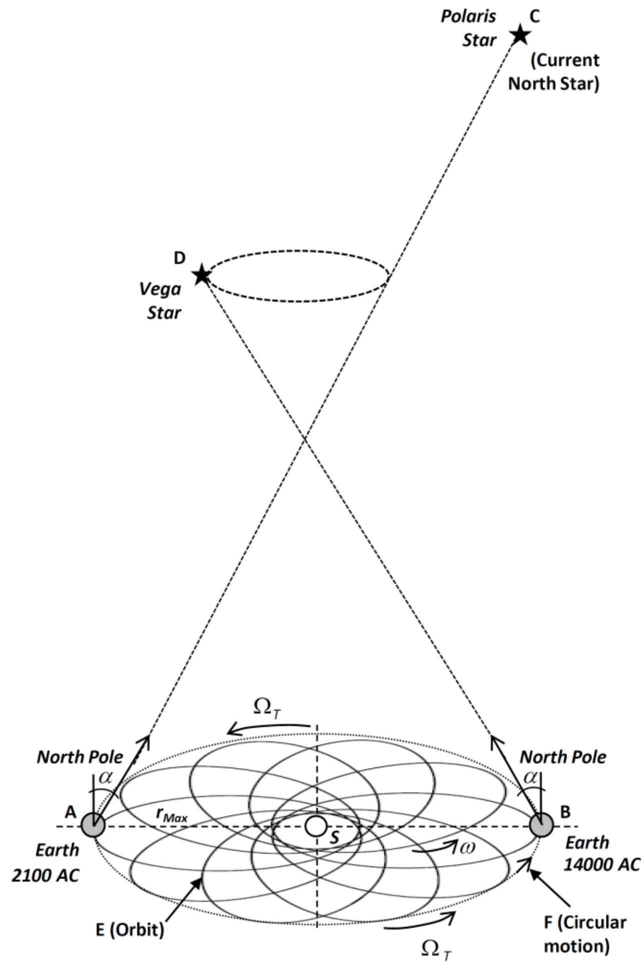

Figure 2. Entire disk of the Solar System scheme rotating like a rigid body for the Earth's orbit. Earth periodically would be changing its position with respect to the considered "fixed" stars

For the maximum Earth-Sun distance r_{Max} and the related minimum orbital velocity, substituting the Earth-Sun system data [28] into (47), the total angular frequency equals 7.72554×10^{-12} radians per second (0.0139689° per year), and for one revolution, the total period is 8.133×10^{11} seconds (25,771.5 years), which is according to the observed period of Earth's axial precession. Angular radius (α) is in average of 23.45° with respect to the ecliptic along the changing of the Earth position. Since a consequence of the axial precession is a changing pole star, we interpret from the Figure 2 that about the year 2100 AC (point A), Earth's North Pole will be appointing near of Polaris star (point C).

Later, when the Earth travels out by half period of the Platonic year to the opposite side (point B), Earth's North Pole will appoint near of Vega star (point D) in the year 14000 AC, changing the Earth its position within the Solar System, and resulting in the effect of the axial precession, coinciding with the known observations and predictions.

5. Rotational Velocities Solution in Spiral Galaxies

We revisit the equation that describes the rotational velocity of stars in spiral galaxies based on the general relativity solution [11]. And based on the Kerr metric, dynamics and geometry of barred spiral galaxies [12] are explained. In particular, we show examples of the calculated rotation curves of unbarred spiral galaxies NGC 4378 and NGC 4594, and barred spiral galaxies Milky Way and NGC 7541, comparing our calculations with the observations, finding a good approximation.

5.1. Kerr Metric in the Barred Spiral Galaxies

The Kerr metric [29] is an exact solution of the vacuum Einstein equations that generalizes to a rotating uncharged black hole with angular velocity $\Omega_\bullet$ different of zero and of massive resting mass $M_\bullet$ of the Schwarzschild metric [24]. In Kerr's original paper, he presented the metric in the following form

$$
\begin{aligned}
ds^2 = &-\left(1-\frac{2mr}{r^2+a^2\cos^2\theta}\right)\left(dv-a\sin^2\theta d\Phi\right)^2 \\
&+2\left(dv-a\sin^2\theta d\Phi\right)\left(dr-a\sin^2\theta d\Phi\right) \\
&+\left(r^2+a^2\cos^2\theta\right)\left(d\theta^2+\sin^2\theta d\Phi^2\right),
\end{aligned}
\tag{48}
$$

where $a = J_\bullet/M_\bullet c$ is the ratio between the angular momentum of rotation $J_\bullet$.

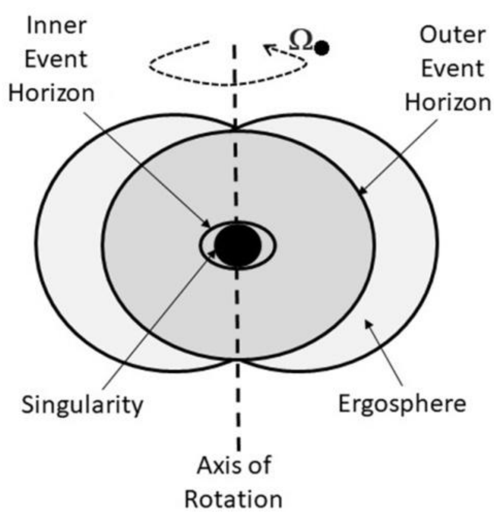

Figure 3. Kerr metric of a rotating black hole

The Schwarzschild curvature singularity at $r = 0$ is replaced in the Kerr metric by $r^2 + a^2\cos^2\theta = 0$, that is, $r = 0$ and $\theta = \pi/2$. Due to the $\cos^2\theta$ term in the square root solution, the outer surface resembles a flattened sphere that touches the inner surface at the poles of the rotation axis, where the colatitude θ equals 0 or π; the space between these two surfaces is called the ergosphere. Thus, the inner surface marks the event horizon [30] (Figure 3). The size and shape of these surfaces of a would depend on the black hole's mass and angular momentum.

Furthermore, a barred spiral galaxy has a central bar that starts from diametrically opposite points of the galactic nucleus. Then, taking into account the Kerr metric, we can consider that stars and gas that are moving at the outer zones with the spiral arms, where the outer event horizon is near of the surface.

When stars and gas reach the region of one of the two rotation poles of the black hole, it is strongly suggested (but does not yet confirmed) that matter and gas clouds interact at the edges of the bar losing angular momentum and thus facilitating the creation of a flow of matter and gas that is diverted following a straight path from the outside region towards the axial poles, then forming the bar (Figure 4).

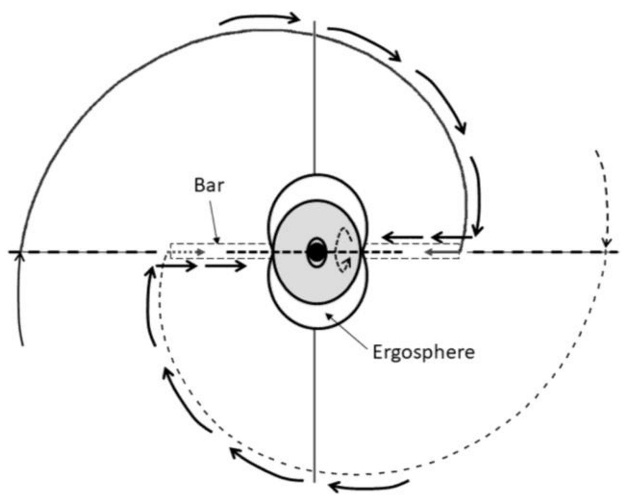

Figure 4. The dynamics around the Kerr metric of a rotating black hole, and the forming of the bar of a barred spiral galaxy

5.2. Rotational Velocity for Spiral Galaxies

Considering Eq. (37) from the relativistic solution for the force F_C, in the same way as with the equation for classical mechanics scenario, we equate it to the centrifugal force F_c [11], obtaining:

$$
\begin{aligned}
&F_C = F_c, \\
&\frac{3GMm\Omega^2}{c^2} = m\frac{v^2}{r}, \\
&v = \sqrt{\frac{3GMr\Omega^2}{c^2}}.
\end{aligned}
\tag{49}
$$

Furthermore, according to Eq. (21), we can write the angular momentum as:

$$J = M_g r^2 \Omega, \tag{50}$$

where M_g is the mass of the system (in this case, mass of the spiral galaxy).

Then, the rotational velocity can be calculated by the angular velocity of the system, by its equivalence with angular momentum J, or by the specific angular momentum $j = J/M_g$ [31], which increases almost linearly with respect to any distance from the nucleus as a function of distance in the spiral galaxies, giving:

$$\begin{gathered} v = \sqrt{\frac{3GM_\bullet}{c^2 r^3}\frac{J^2}{M_g^2}}, \\ v = \sqrt{\frac{3GM_\bullet j^2}{c^2 r^3}}, \end{gathered} \tag{51}$$

with M in this case being the resting mass $M_\bullet$ of the galactic nucleus. Having that $\theta = \Omega t$ and, for the sake of describing the geometry of this equation, considering the Schwarzschild radius r_s [24]. Then, we write (49) in polar coordinates as:

$$\begin{gathered} \frac{2v}{3}\left(\frac{r}{t}\right) = \frac{2GM_\bullet r\Omega}{c^2}\left(\frac{v}{r}\right), \\ r = \pm\frac{3}{2} r_s \theta, \end{gathered} \tag{52}$$

which is the equation of a spiral. The two main arms of a spiral galaxy can be geometrically represented in (52) by considering the (±) signs. In addition, considering the Kerr metric, geometry around a rotating axially-symmetric black hole would form the bar from the spiral towards the axial poles. Thus, from (52), the geometry of barred spiral galaxies can be written as:

$$r = \pm\frac{3}{2} r_s^{\theta}, \tag{53}$$

which is the equation of a spiral with two centres (Figure 4). The length of the bar will depend on how far the effect of the horizon event at the rotation poles reaches to transport matter and gas from the outer region towards the singularity. Following the Kerr metric, it depends on the mass of the black hole, the rotation speed, and the closeness that the outer horizon event reaches to the external environment at the rotation poles.

Table 1. Rotational velocities: Tabulation of respective rotational velocities calculated from Eq. (51) for the unbarred spiral galaxies: NGC 4378 and NGC 4594, and the barred spiral galaxies: Milky Way and, NGC 7541

Galaxy	Sa NGC 4378			Sa NGC 4594			SB Milky way			SB NGC 7541		
Central mass ($M_\bullet$)	7.9×10^7 M$_\odot$			6.6×10^8 M$_\odot$			2×10^{10} M$_\odot$			4.9×10^{10} M$_\odot$		
Total mass (M_g)	3.65×10^{11} M$_\odot$			3.66×10^{11} M$_\odot$			1.3×10^{12} M$_\odot$			4.6×10^{11} M$_\odot$		
Distance from nucleus (kps)	Log (J)	J (kg m^2 s^{-1})	Rotational velocity (km s^{-1})	Log(J)	J (kg m^2 s^{-1})	Rotational velocity (km s^{-1})	Log (J)	J (kg m^2 s^{-1})	Rotational velocity (km s^{-1})	Log (J)	J (kg m^2 s^{-1})	Rotational velocity (km s^{-1})
0	0	0	0	0	0	0	0	0	0	0	0	0
1.25	70.91	8.21×10^{70}	280	70.09	1.22×10^{70}	120	87.31	2.03×10^{87}	249	69.38	2.38×10^{69}	160
2.5	71.42	2.64×10^{71}	318	70.82	6.63×10^{70}	230	87.74	5.47×10^{87}	238	69.86	7.24×10^{69}	172
5	71.87	7.41×10^{71}	316	71.40	2.49×10^{71}	305	88.12	1.33×10^{88}	204	70.31	2.03×10^{70}	170
7.5	72.12	1.32×10^{72}	306	71.69	4.85×10^{71}	324	88.45	2.81×10^{88}	235	70.55	3.58×10^{70}	164
10	72.30	1.99×10^{72}	300	71.88	7.66×10^{71}	332	88.59	3.92×10^{88}	213	70.74	5.53×10^{70}	164
12.5	72.44	2.73×10^{72}	295	72.04	1.09×10^{72}	339	88.80	6.25×10^{88}	243	70.89	7.84×10^{70}	167
15	72.55	3.56×10^{72}	292	72.16	1.45×10^{72}	343	88.91	8.11×10^{88}	240	71.02	1.05×10^{71}	170
20	72.73	5.37×10^{72}	286	-	-	-	89.11	1.29×10^{89}	247	71.23	1.69×10^{71}	178
22	72.79	6.19×10^{72}	285	-	-	-	89.17	1.49×10^{89}	249	71.30	2.00×10^{71}	182
25	-	-	-	-	-	-	89.26	1.82×10^{89}	251	71.39	2.47×10^{71}	186

5.3. Comparison between Calculated and Observed Rotational Velocities in Spiral Galaxies

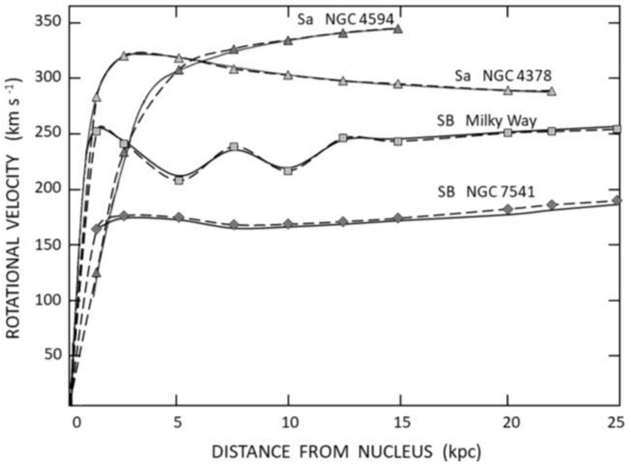

Figure 5. Plot of respective rotational velocities versus distance from nucleus for the unbarred spiral galaxies: NGC 4378 and NGC 4594, and the barred spiral galaxies: Milky Way and, NGC 7541. The dashed lines are calculated from Eq. (51), and the solid lines come from the observations

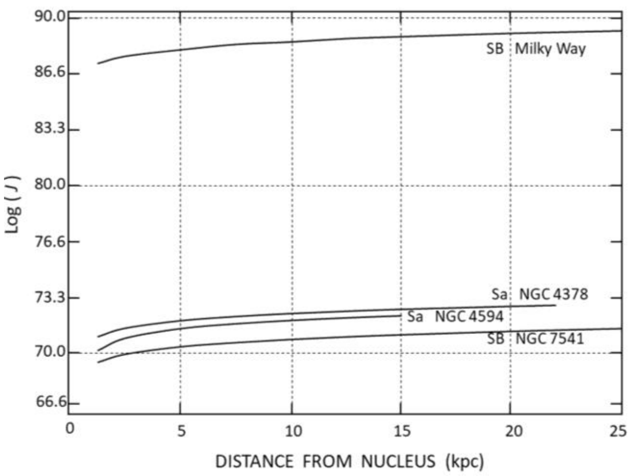

Figure 6. Plot of respective angular momentum for the unbarred spiral galaxies: NGC 4378 and NGC 4594, and the barred spiral galaxies: Milky Way and, NGC 7541. The curves show the same almost linear pattern for all these spiral galaxies along the different distances from their respective nucleus

The rotational velocity of some unbarred and barred spiral galaxies are calculated by applying (51). In particular, considering as an example the known rotation curves of the spiral Sa galaxies: NGC 4378 and NGC 4594, and spiral SB galaxies: Milky Way and, NGC 7541. Thus, having the known values for each of these four spiral galaxies [32-38], the results are tabulated in Table 1 (Rotational velocities) for each galaxy. It is observed that the nucleus masses of barred spiral galaxies are larger than the nucleus masses of unbarred spiral galaxies. The rotation curves are plotted in Figure 5, which shows the calculated results for these galaxies. The observed rotation curves of these galaxies [39,40] are superimposed on the calculated rotation curves. Then, the solid lines in Figure 5 show the rotation curves obtained from the observations. And the rotation curves calculated with (51) are shown as dashed lines, finding a good approximation between the calculated rotation curves and the known rotation curves of these spiral galaxies. The respective angular momentum along the different distances from the nucleus for these spiral galaxies are plotted in Figure 6. It is shown the same almost linear pattern of angular momentum for all these spiral galaxies. On the other hand, the Oort constants [41] have not been considered in this work, since such constants are empirically derived parameters that particularly characterize the local rotational properties of the Milky Way.

6. Conclusions

This work aims to summarize the previous works that together describe the theory in which the movement of some circular or disk cosmological systems, such as the entire disk of the Solar System and spiral galaxies, corresponds to the dynamics like a rigid body in rotational motion. It has been found that both the Lagrangian solution and the relativistic solution consider all the energies and forces involved in the rotating systems in the three-dimensional framework. In this way, most of the equations on this work are derived from Eq. (1), which is the formula of the system for the rigid body in rotational motion. This shows that the relativistic solution for the total force considers the dynamics of a rotating system like a rigid body in rotational motion. There are derived the respective equations to explain the apsidal precession of the planets, the Earth’s axial precession and the rotation curves of some spiral galaxies, all of them due to the effect of rotating systems like a rigid body in rotational motion. A good approximation between the calculated and the observations for each of these cases has been found. Thus, this theory unifies classical mechanics (through Lagrangian mechanics) with the solution of general relativity in the three-dimensional framework. Also, it unifies all these rotational behaviour of these circular or disk cosmological systems in the same concept of the rigid body in rotational motion. Then, (51) describes the rotational velocity of some spiral galaxies based on this relativistic solution. We present preliminarily examples of the rotation curves of unbarred spiral galaxies: NGC 4378 and NGC 4594, and barred spiral galaxies: Milky Way and, NGC 7541. Comparing our calculations with the observations we find a good approximation. Furthermore, the existence of barred spiral galaxies could be a confirmation of the Kerr metric. It is shown that the respective angular momentum for these spiral galaxies has the same almost linear pattern along the different distances from their respective nucleus. One of the significances of this result based on the general relativity solution is that it is possible to fit the known rotation curves of the spiral galaxies without any need of introducing dark matter at all. The next step in proving the galaxy dynamics on the general relativity solution is to make more detailed observations of their angular momentum to confirm if the way in which spiral galaxies rotate is mainly according to rigidly rotating and long-lived patterns, as steady spirals.

ACKNOWLEDGEMENTS

The author would like to thank Professor Sergio S. Cornejo for its review and comments for this work.

REFERENCES

[1] Newton, I., Philosophiæ Naturalis Principia Mathematica, (Cambridge, UK, 1867), London (1982).

[2] Unzicker, A., 2008, Why do we still believe in Newton's law? Facts, Myths and Methods in Gravitational Physics. arXiv:gr-qc/0702009v8.

[3] Sofue, Y., and Rubin, V. C., 2001, Astron. Astrophys. Vol. 39, p.137. arXiv: astro-ph/0010594.

[4] Milgrom, M., 1983, A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis. Astrophysical Journal. 270: 365–370. DOI: 10.1086/161132.

[5] Kyu-Hyun Chae et al., 2020, Testing the strong equivalence principle: Detection of the external field effect in rotationally supported galaxies, Ap J 904: 51, https://doi.org/10.3847/1538-4357/abbb96.

[6] Einstein, A., 1955, The Meaning of Relativity, 5th ed. Princeton University Press, Princeton.

[7] Arbab, A. I., 2012, J. Mod. Phys., Vol. 3 No. 29, 1231-1235. arXiv:1105.1911 [physics.gen-ph].

[8] Cornejo, A. G., 2014, A lagrangian solution for the precession of Mercury's perihelion, Int. J. Astron. 3, 31–34. http://article.sapub.org/10.5923.j.astronomy.20140302.01.html.

[9] Cornejo, A. G., 2013, The rotating reference frame and the precession of the equinoxes, Lat. Am. J. Phys. Educ. 7(4), 591. http://www.lajpe.org/dec13/11-LAJPE_834_Adrian_Cornejo.pdf.

[10] Cornejo, A. G., 2021, Axial precession in the general theory of relativity solution, Int. J. Astron. 10(1): 1-5. http://article.sapub.org/10.5923.j.astronomy.20211001.01.html.

[11] Cornejo, A. G., 2020, The rotational velocity of spiral Sa galaxies in the general theory of relativity solution, Int. J. Astron. 9(2): 27-30. http://article.sapub.org/10.5923.j.astronomy.20200902.01.html.

[12] Cornejo, A. G., 2021, The rotational velocity of barred spiral galaxies in the general relativity solution, Int. J. Astron. 10(1): 6-12. http://article.sapub.org/10.5923.j.astronomy.20211001.02.html.

[13] Baba, J., Morokuma-Matsui, K., Miyamoto, Y., Egusa, F., and Kuno, N., 2016, Gas velocity patterns in simulated galaxies: observational diagnostics of spiral structure theories. Monthly Notices of the Royal Astronomical Society, 460(3), 2472-2481.

[14] Vladimir, I. A., 1989, Mathematical Methods of Classical Mechanics, 2nd edn. Springer, p. 130.

[15] Tocaci, E., 1984, Relativistic Mechanics, Time, and Inertia, C. W. Kilmister Ed., Cambridge University Press.

[16] Hand, L. N. and Finch, J. D., 2008, Analytical Mechanics, Cambridge: Cambridge University Press.

[17] Ramos, J., de Montigny, M. and Khanna, F. C., 2010, Gen. Relativ. Gravit. 42(10).

[18] Goldstein, H., Poole, Ch. P., and Safko, J. L., 2002, Classical Mechanics, 3rd edn., Addison-Wesley Publishing Company, Inc. Reading, Massachusetts.

[19] Tan, A., 2008, Theory of Orbital Motion, Singapore: World Scientific Publishing Co. Pte. Ltd.

[20] Danby, J. M. A., 1988, Fundamentals of Celestial Mechanics. 2nd edn., Richmond: Willmann-Bell, Inc.

[21] Weinberg, S., 1972, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. John Wiley. ISBN 978-0-471-92567-5.

[22] Cheng, T., 2005, Relativity, gravitation, and cosmology, Oxford University Press. pp. 98–99.

[23] Cornejo, A. G., 2010, The equivalent expressions between escape velocity and orbital velocity, Lat. Am. J. Phys. Educ., 4(3), p. 578. http://www.lajpe.org/sep10/421_Adrian_Cornejo.pdf.

[24] Schwarzschild, K., 1916, Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. Sitzungsberichte der Königlichen Preußischen Akademie der Wissenschaften zu Berlin 189–196.

[25] Scarborough, J. B., 1958, The Gyroscope: Theory and Applications. New York: Interscience Publishers, Inc., pp. 37-52.

[26] Einstein, A., 1915, Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie. Königlich Preußischen Akademie der Wissenschaften (Berlin) 47(2) pp. 831-839.

[27] Beatty, J. K., Petersen, C. C., Chaikin, A., Chaikin, A. L., 1990, The New Solar System, 4th edn. Cambridge University Press, Cambridge.

[28] Williams, D. R., 2019, Planetary Fact Sheet – Metric, NASA, Houston, Texas. http://nssdc.gsfc.nasa.gov/planetary/factsheet. Accessed on June 9th, 2020.

[29] Kerr, R. P., 1963, Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics. Physical Review Letters. 11 (5): 237–238. DOI:https://doi.org/10.1103/PhysRevLett.11.237.

[30] Visser, M., 2008, The Kerr spacetime: A brief introduction, arXiv:0706.0622.

[31] Romeo, A. B., and Mogotsi, K. M., 2018, Mon. Not. R. Astron. Soc. 000, 1–5. arXiv:1805.05871 [astro-ph.GA].

[32] Akylas, A., and Georgantopoulos, I., 2009, Astron. Astrophys. 500.3: 999–1012. DOI:http://dx.doi.org/10.1051/0004-6361/200811371.

[33] Jardel, J. R. et al., 2011, Orbit-based dynamical model of the sombrero galaxy (NGC 4594). ApJ: 30. arXiv:1107.1238.

[34] Rubin, V. C., Ford, W. K. Jr., Strom, K. S., Strom, S. E., and

Romanishin, W., 1978, ApJ, Vol. 224, p. 782-795.

[35] McMillan, P. J., 2016, The mass distribution and gravitational potential of the Milky Way, Mon. Not. R. Astron. Soc. 419, 2095–2115. arXiv:1608.00971.

[36] Valenti, E. et al., 2016, Stellar density profile and mass of the Milky Way bulge from VVV data, A&A 587, L6. DOI: https://doi.org/10.1051/0004-6361/201527500.

[37] Kyazumov, G. A., 1980, Velocity field of the galaxy NGC 7541, Pisma v Astronomicheskii Zhurnal, vol. 6, July 1980, p. 398-401. Soviet Astronomy Letters, vol. 6, July-Aug. 1980, p. 220-222. Translation.

[38] Mineo, S., Gilfanov, M., and Sunyaev, R. 2012, X-ray emission from star-forming galaxies – I. High-mass X-ray binaries, Mon. Not. R. Astron. Soc. 419, 2095–2115. DOI: https://doi.org/10.1111/j.1365-2966.2011.19862.x.

[39] Rubin, V. C., Ford, W. K. Jr., and Thonnard, N., 1978, ApJ, 225: L107-L111.

[40] Kutner, M. L., 2003, Astronomy: A Physics Perspective. Cambridge University Press, Cambridge, 2nd ed. DOI: https://doi.org/10.1017/CBO9780511802195.

[41] Binney, J., and Tremaine, S., 2008, Galactic Dynamics, 2nd ed. Princeton University Press, Princeton.