

# Quantum Theory of Synchrotron Radiation

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**Abstract** A previous quantum mechanical description of synchrotron radiation led to unphysical results. The reasons for this error were limitations in the Erdelyi- Swanson asymptotic approximation of the Whittaker functions. These problems were rectified with a new derivation of the asymptotic expansion. The most prominent feature of the new synchrotron spectrum is that under extreme conditions the maximum radiation is shifted upwards towards the tip.

**Keywords** Magnetor, Quantum, Synchrrotron, Asymptotic

## 1. Overview

With current equipment it is possible to accelerate electrons to energies of the order of  $10^{12}$  eV. Also with high energy condenser banks and spiral generators pulsed magnetic fields in the range 1-10 MG (1 MG =  $10^6$  Gauss) can be produced.<sup>1</sup>

Even more extreme conditions occur in astronomy. Electrons with energies of  $10^{18}$  eV have been detected in cosmic ray showers, and it is surmised that 'magnetars' can sustain magnetic fields up to  $10^{15}$  Gauss. Under these circumstances the classical computation of synchrotron radiation leads to unphysical results. This error is due to identifying the synchrotron intensity spectrum with the Fourier resolution of the classical radiation field. Under extreme conditions the upper end of the radiation field includes photons whose energies exceed the energies of the incident electrons. Since the classical computations are not constrained by energy conservation this violation leads to unphysical results. Ironically, a previous attempt to resolve these issues with a complete quantum mechanical computation also led to unphysical results [1]. The general scheme of the quantum calculation is shown in Fig. 2. This depicts the situation where an electron in an initial state  $\psi_i$  with energy  $E_i$  and momentum  $\vec{p}_i$  emits a photon  $\gamma$  with energy  $\hbar\omega$  and momentum  $\vec{k}$  while transitioning to a final state  $\psi_f$  with energy  $E_f$  and momentum  $\vec{p}_f$ . In quantum mechanics energy and momentum conservation are built in from the outset:

$$\begin{aligned}\hbar\omega &= E_i - E_f \\ \vec{k} &= \vec{p}_i - \vec{p}_f\end{aligned}\quad (1)$$

However, these constraints do not determine unique final states. Moreover, arbitrarily large momentum transfers can be absorbed by the magnetic fields: in principle this momentum is ultimately taken up by the macroscopic devices that generate the fields. The stochastic features enter in the overlap integral  $\langle \psi_i | \exp(i\vec{k} \cdot \vec{r}) | \psi_f \rangle$  which, as indicated on Fig. 2, effectively determines the shape of the synchrotron radiation spectrum. The explicit evaluation of the overlap integral is given in Ref. [1]. The results are lengthy expressions whose essential components are the generalized Laguerre functions  $\mathcal{L}_n^\alpha(\kappa)$  whose argument ( $\kappa$ ), index ( $n$ ), and order ( $\alpha$ ) are given by

$$\begin{aligned}\kappa &= 4.45 \times 10^7 (\hbar\omega)^2 (\text{MeV}) / H(\text{MG}) \rightarrow 4.2 \times 10^{12} \\ n &= 8.45 \times 10^{13} E^2 (\text{GeV}) / H(\text{MG}) \rightarrow 4.2 \times 10^{20} \\ \alpha &= 1.69 \times 10^{11} (\hbar\omega)(\text{MeV}) E(\text{GeV}) / H(\text{MG}) \rightarrow 8.5 \times 10^{16}\end{aligned}\quad (2)$$

The special values correspond to the physical parameters  $E \sim 5$  TeV,  $H \sim 5$  MG, and photon energies  $\hbar\omega \sim 500$  MeV. Clearly, a direct numerical evaluation of the Laguerre functions is not feasible and it is necessary to resort to asymptotic expansions. In Ref. [1] this next step utilized the expansions developed by Erdelyi and Swanson Ref. [2]. Unfortunately the end results also turned out to give unphysical predictions for the synchrotron radiation spectrum. A line-by-line check of the computations in [1] showed that errors occurred in the Erdelyi-Swanson asymptotic expansions. The situation is illustrated graphically in Fig. 3. The plots show the behavior of the Whittaker function  $M_{n,\alpha}(z)$  — which is closely related to the Laguerre function

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<sup>1</sup> The kinematics of the synchrotron radiation emitted when electrons traverse a magnetic field are shown in Fig. 1.

— for modest values of the index and order;  $n=11$  and  $\alpha=15/2$ . Obviously, even for these small parameter values there is no useful resemblance between the Erdelyi-Swanson asymptotic expansion  $\hat{M}$  and the exact result  $M$ . Overcoming these difficulties required the development of new asymptotic expansions for the Whittaker functions,

including rigorous error bounds [3]. Fig. 3 shows that the new expansion  $\tilde{M}$  is a close fit to the exact result  $M$ , and this improvement also holds for large parameter values such as those in Eq.(2). The complete calculation of the overlap integrals then yields new results for the synchrotron radiation spectrum.

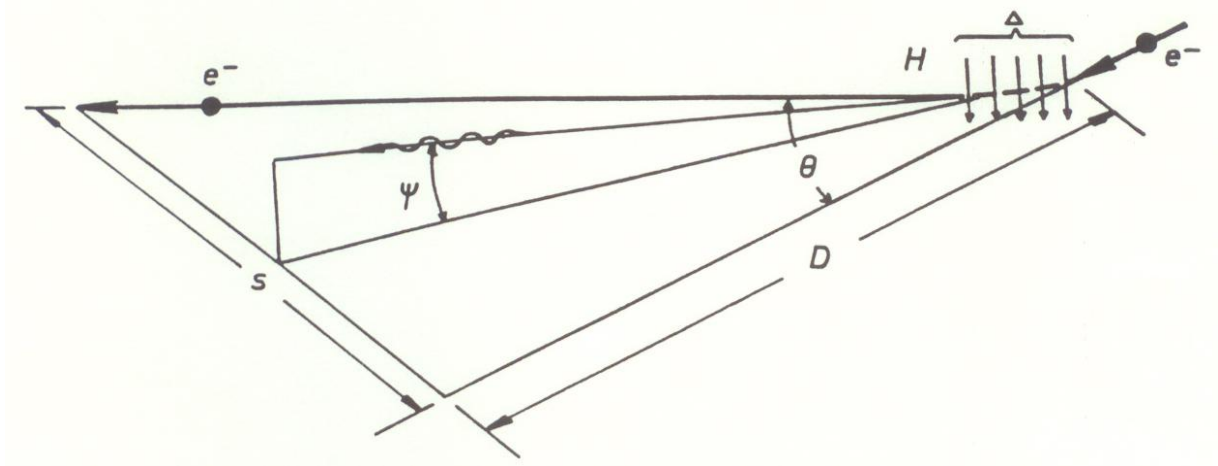


Figure 1. Kinematics of magnetic bremsstrahlung

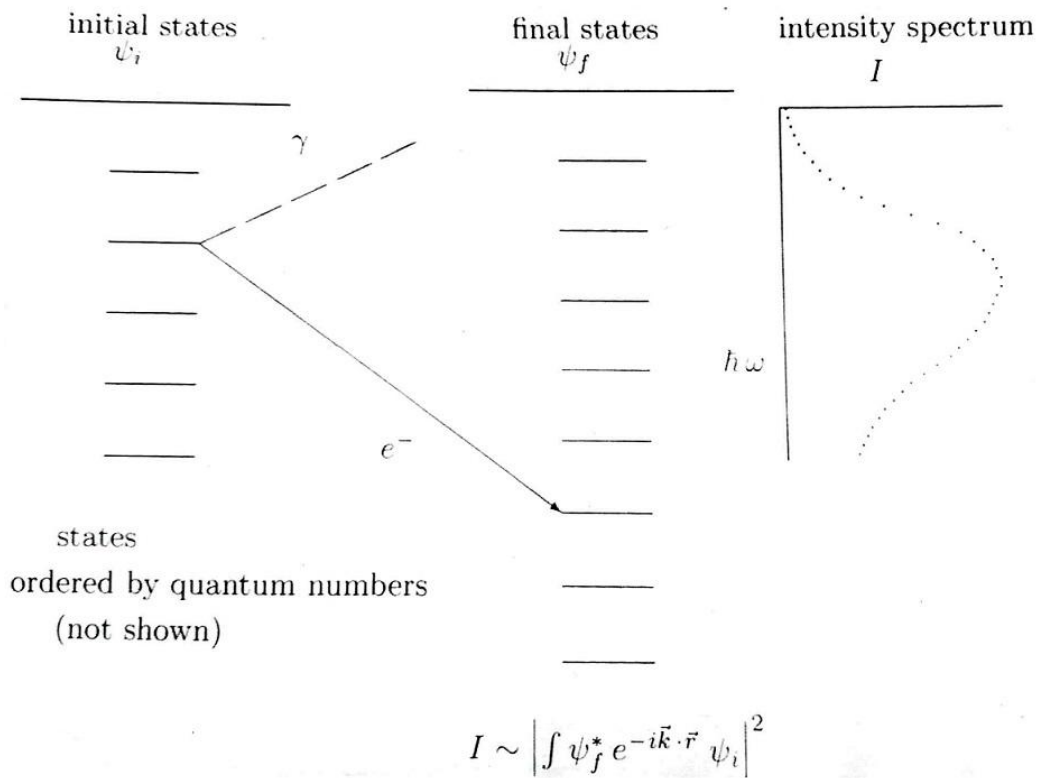


Figure 2. Schematic for Photon Emission

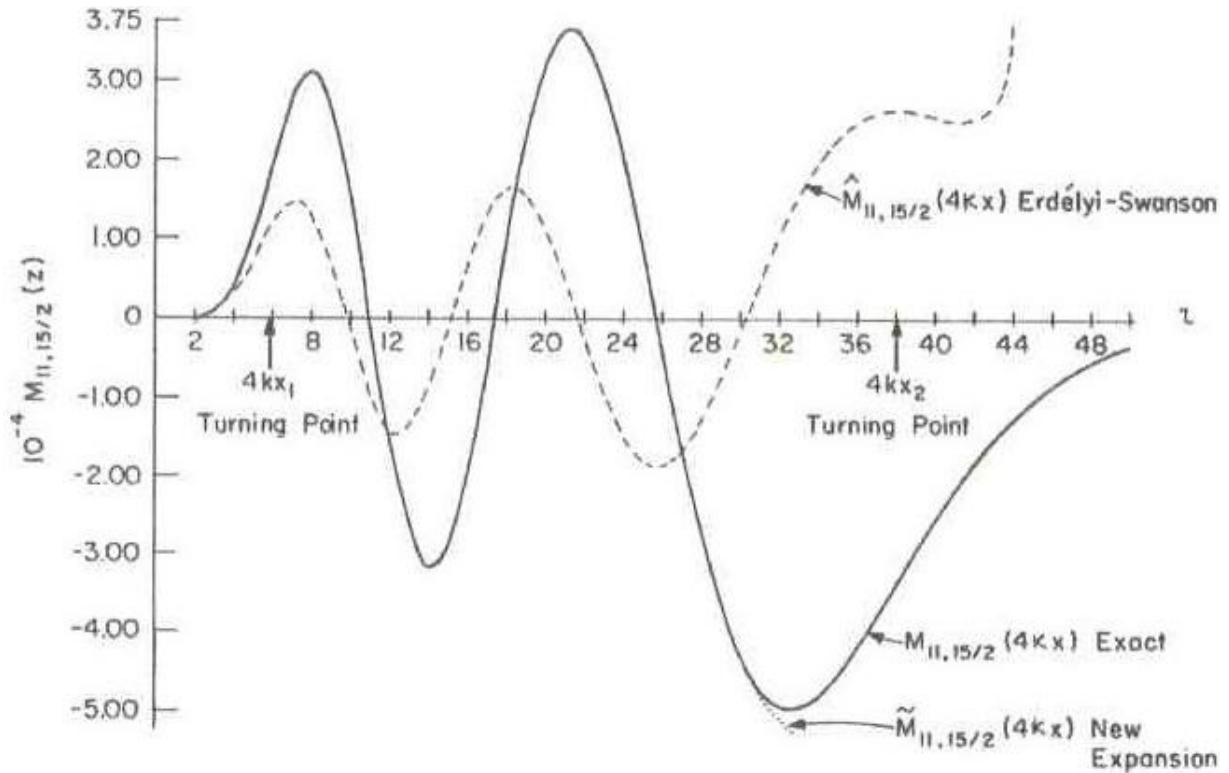


Figure 3. Comparison of the exact value of  $M_{11,15/2}$  with the new and Erdelyi-Swanson Asymptotic Expression

## 2. The Synchrotron Radiation Spectrum

The detailed derivation of the uniform Whittaker asymptotic expansion is given in Ref. [3]. Here it is sufficient to note that the essential idea is to use sequential or telescoping expansions. Specifically

$$M_{\kappa,m}(4\kappa z) \rightarrow J_{2m}(2\kappa\zeta^{1/2}) \rightarrow Ai(2m^{2/3}\zeta) \quad (3)$$

The argument of the Airy function contains the factor  $\zeta$ .

$$\begin{aligned} \frac{2}{3}(\zeta)^{3/2} &= \ln[1 + \sqrt{1-z^2}] - \ln z - \sqrt{1-z^2} & 0 \leq z \leq 1 \\ \frac{2}{3}(-\zeta)^{3/2} &= \sqrt{z^2-1} - \sec^{-1}(z) & 1 \leq z \leq \infty \end{aligned} \quad (4)$$

The auxiliary variable  $z$  is determined by the argument of the Whittaker function. In principle this ‘double descent’ can be extended another step because the Airy function can be approximated by

$$Ai(y) \sim \frac{\exp(-\frac{2}{3}y^{3/2})}{2\sqrt{\pi}y^{1/4}}. \quad (5)$$

Extensive tabulations of the Airy function are also available. Fig. 4 gives an indication of the trend of the results by comparing the classical and quantum mechanical radiation rates as functions of the dimensionless parameter

$$Y = \frac{E}{mc^2} \frac{H}{H_{cr}} \approx 4.43 \times 10^{15} E(\text{GeV}) H(\text{MG}) \quad (6)$$

The quantum features enter the calculation through the scale of the magnetic field

$$H_{cr} = \frac{m^2 c^3}{e\hbar} \approx 4.414 \times 10^{13} \text{G} \quad (7)$$

Earlier work attempted to gauge the deviations from the classical results by successive expansions in the quantum parameter  $Y$ . As shown in Fig. 4 these computations were confused by the fact that the second order  $Y^2$  corrections appeared to dominate the first order  $Y$  corrections. Clarity was eventually achieved by comparison with the ‘exact’ results, where ‘exact’ refers to spectral calculations utilizing the new asymptotic expansions.

Fig. 5 shows the upper end of the synchrotron spectrum for fairly demanding technical conditions; i.e.  $E \sim 800$  GeV,  $H \sim 2$  MG, and  $Y \sim 0.07$ . Evidently most of the classical result falls below 300 GeV and so there is no significant violation of energy conservation. The plot shows that it would be very difficult to distinguish between the classical and quantum mechanical spectral shapes.

Fig. 6 displays the situation for still more extreme — but technically feasible — parameter values  $E \sim 5$  TeV,  $H \sim 3.6$  MG, and  $Y \sim 0.8$ . Now a significant portion of the classical spectrum extends above the energy conservation limit  $\hbar\omega = E$ , and accordingly the classical predictions have to be discarded as unphysical. By contrast, the quantum mechanical spectrum ( $I_Q$ ) cuts off at  $\hbar\omega = E$  as it should. Furthermore, the maximum of the radiation rate has shifted upwards towards the tip.

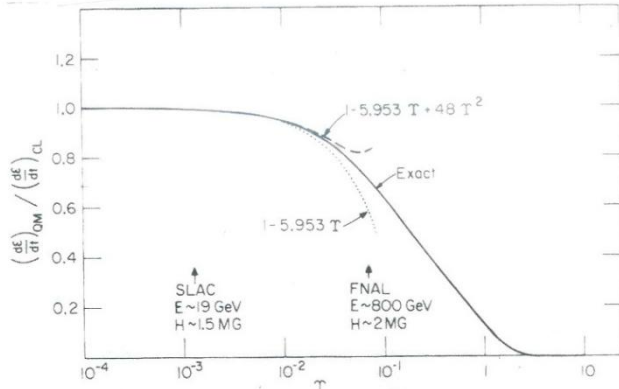


Figure 4. Aftermaths of SLAC

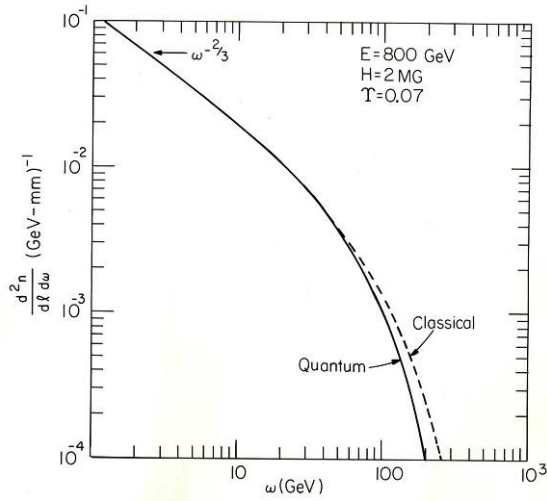


Figure 5. Classical versus quantum radiation spectrum

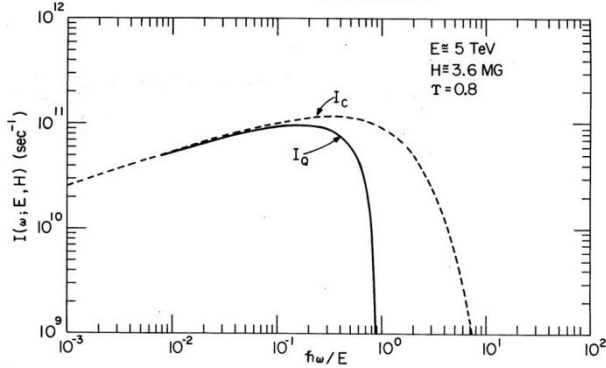


Figure 6. Classical versus quantum spectra

The character of the radiation is altered even more drastically in the extreme quantum regime.

Fig. 7 shows that when  $Y \sim 10^3$ , which may occur in magnetars, the quantum radiation spectrum has increased monotonically with energy to a maximum at the tip  $\hbar\omega \leq E$ .

An overview of the various field and energy combinations determining the values of  $Y$  is given in Fig 8. The 'VBA' point refers to the parameter values associated with a 'Very Big Accelerator'. To date this project has been approximated by the construction of the 'Large Hadron Collider' ( $E \sim 6$  TeV) at CERN. Due to their speculative status magnetars

have been omitted from the diagram Ref. [4].

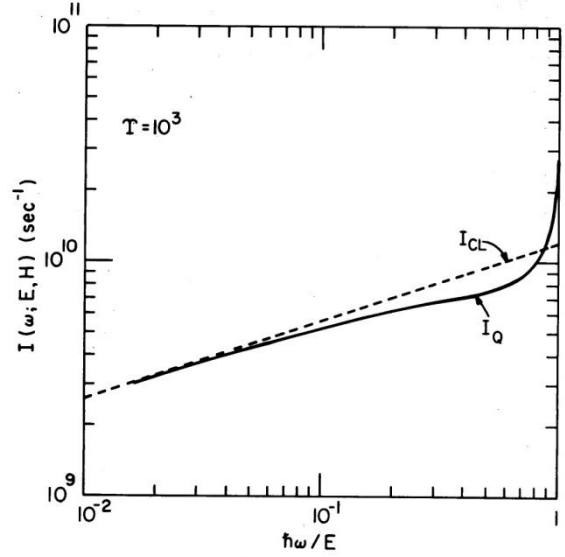


Figure 7. Classical versus quantum radiation spectrum

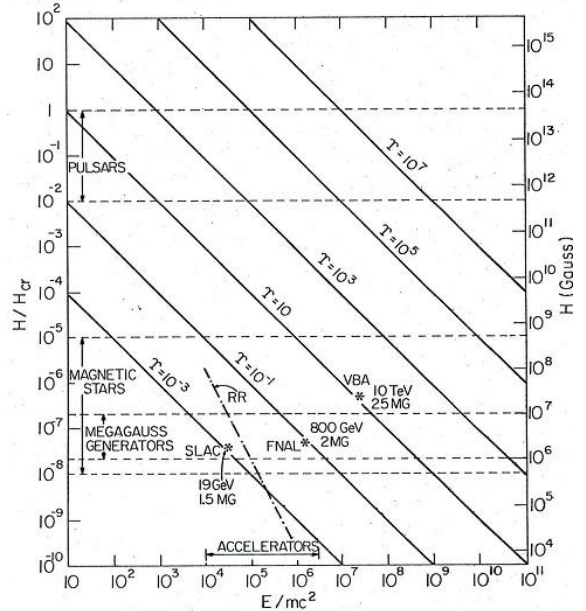


Figure 8. Overview

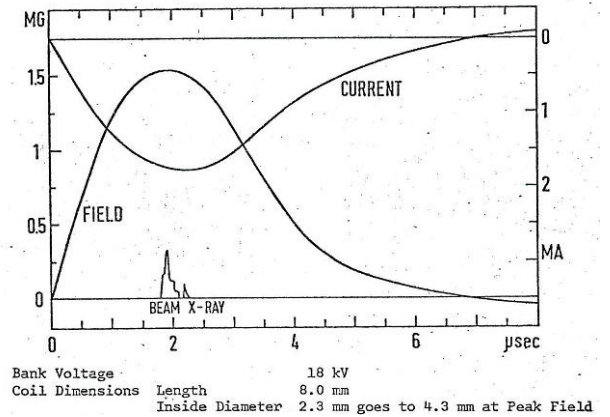
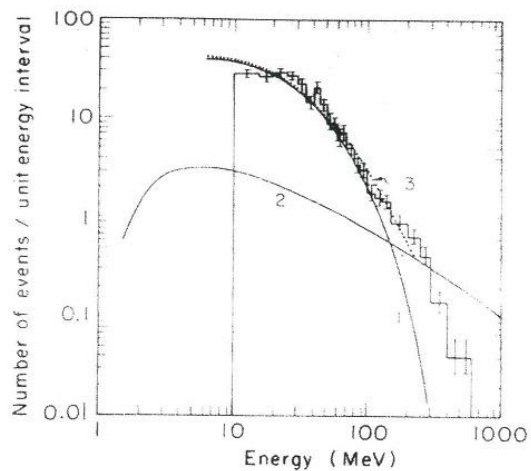


Figure 9. Timing Sequence



**Figure 10.** 1. Theoretical prediction, 2. Beam Halo, 3. Measured Spectrum

Synchrotron radiation is widely used in medicine and in industrial fabrication; but the SLAC point at 19 GeV, 1.5 MG, and  $Y \sim 10^{-3}$  represents the only demonstration of the process under robust conditions. The details of this experiment are described in Refs. [5] and [6]. The essential point is to shoot a single compact beam pulse of width 1.3 mm containing  $10^6$  electrons through a coil with 2 mm entrance and exit holes.

Fig. 9 shows that the timing sequence is arranged so that the beam pulse passes through the coil just as the magnetic field reaches its peak value of 1.5 MG. The resultant photon spray is incident on an Ilford G-5 nuclear emulsion target where about 0.5% of the photons convert into  $e^+e^-$  pairs.

Microscopic measurements of the multiple scattering of the pairs in the emulsion can then be calibrated to yield the incident photon energies Ref. [7]. The final results for the synchrotron spectrum are shown in Fig. 10. There is satisfactory agreement between experiment and theory.

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