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Symmetries and Exact Solutions of Some Partial Differential Equations

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Abstract Sophus Lie developed technique to obtain solutions of differential equations using continuous symmetries. Using these continuous symmetries, Peter E. Hydon developed technique to obtain discrete symmetries which led to finding further new solutions of the underlying equations. In this paper continuous and discrete symmetries of *Korteweg de Vries* and *nonlinear filtration equations* are analyzed. Using these symmetries group invariant solutions and the exact solutions of these equations are obtained.

Keywords Continuous symmetries, Discrete symmetries, Group invariant solutions

1. Introduction

Differential equations is an important branch of mathematics. Since the time of Leibneiz and Newton, different attempts have been made to develop techniques for finding their solutions. The process is still going on.

In this context, Sophus Lie developed a technique known as group theoretic/symmetry methods for finding the solutions of differential equations. His method is, though, highly algorithmic but has an advantage that if the symmetry of that differential equation exists then it either gives the solution or reduce the problem into a comparatively simpler one. The symmetries used in his methods are continuous i.e. they depend upon some parameter [1]-[3]. In contrast to continuous symmetries there are other symmetries known as discrete symmetries which do not depend on parameter. However, these discrete symmetries are also useful in finding solutions of differential equations.

Using continuous symmetries, Peter E. Hydon developed a technique [4]-[8] to get all discrete symmetries of differential equations. His technique has been used here to get all discrete symmetries of nonlinear filtration equation. Exact solutions, using these symmetries are then presented.

Hydon considered an automorphism of $r-$dimensional Lie algebra L of symmetries of differential equations. This gives a change in basis vectors and is represented as

$$\mathbf{X}_i = b_i^l \mathbf{X}_l. \tag{1}$$

As the structure constants do change under a transformation of the basis, we have the following transformation law

$$c_{lm}^n b_i^l b_j^m = c_{ij}^k b_k^n, i, j, k, l, m, n = 1, \cdots, r\,. \tag{2}$$

These constraints, called the nonlinear constraints. provide a real-valued matrix $B = (b_i^l)$ that corresponds to the automorphism. They fix some of the entries of matrix B. The adjoint action of each basis vector, $\boldsymbol{X}_j$, generates a one-parameter Lie group of inner automorphisms whose matrix representation is [4]

$$A(j,\epsilon) = e^{\epsilon C(j)}$$
$$= I + \frac{\epsilon}{1!}(C(j)) + \frac{\epsilon^2}{2!}(C(j))^2 + \frac{\epsilon^3}{3!}(C(j))^3 + \ldots \tag{3}$$

where

$$(C(j))_i^k = c_{ij}^k. \tag{4}$$

The matrix B can be further simplified with the help of

$$\hat{B} = BA(2,\epsilon_2)A(1,\epsilon_1)A(3,\epsilon_3)A(5,\epsilon_5)A(4,\epsilon_4). \tag{5}$$

The simplified matrix is then used to write determining equation whose solution is of the form

$$(\hat{x}, \hat{t}, \hat{u}) = (\hat{x}(x,t,u), \hat{t}(x,t,u), \hat{u}(x,t,u)) \tag{6}$$

from where we get all the symmetries, i.e. both continuous and discrete.

2. Symmetries of Korteweg de Vries Equation

Korteweg de Vries (KdV) equation

$$\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0, \tag{7}$$

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is a mathematical model of waves on shallow water surfaces. Many different variations of the KdV equation have been studied. The mathematical theory behind the KdV equation is a topic of active research. KdV equation was first introduced by Boussinesq (1877) and rediscovered by Diederik Korteweg and Gustav de Vries (1895).

Following are the basis elements and nonzero structure constants of the 4-dimensional Lie algebra of KdV equation (7)

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \mathbf{X}_2 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u},$$

$$\mathbf{X}_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \mathbf{X}_4 = \frac{\partial}{\partial t}. \quad (8)$$

$$c_{12}^1 = 1, c_{23}^3 = 2, c_{34}^1 = 1, c_{24}^4 = -3. \quad (9)$$

The corresponding (one-parameter) Lie groups of point symmetries

$$e^{\epsilon_1 \mathbf{X}_1}\ (\hat{x}, \hat{t}, \hat{u}) = (x + \epsilon_1, t, u), \quad (10)$$

$$e^{\epsilon_2 \mathbf{X}_2}\ (\hat{x}, \hat{t}, \hat{u}) = (e^{\epsilon_2}x, e^{3\epsilon_2}t, e^{-2\epsilon_2}u), \quad (11)$$

$$e^{\epsilon_3 \mathbf{X}_3}\ (\hat{x}, \hat{t}, \hat{u}) = (x + t\epsilon_3, t, u + \epsilon_3), \quad (12)$$

$$e^{\epsilon_4 \mathbf{X}_4}\ (\hat{x}, \hat{t}, \hat{u}) = (x, t + \epsilon_4, u). \quad (13)$$

The **only** discrete point symmetry of the KdV equation (7) is

$$\Gamma_D\ (\hat{x}, \hat{t}, \hat{u}) = (\,-x, -t, u\,), \quad (14)$$

which can easily be obtained by following theprocedure given in the previous section.

Using the continuous symmetry generators, the one-dimensional optimal algebra is obtained as:

$$\mathbf{X}_1,\ \mathbf{X}_2,\ \mathbf{X}_3,\ \mathbf{X}_4,\ \mathbf{X}_3\text{+}\mathbf{X}_4, \text{ and } \mathbf{X}_3\text{-}\mathbf{X}_4.$$

The discrete symmetry Γ_D, maps $\mathbf{X}_3$-$\mathbf{X}_4$ to $\mathbf{X}_3$+$\mathbf{X}_4$. Thus, there is a reduction of number of elements of the optimal algebra [3].

3. Exact Solutions

In this section exact solution of KdV equation, corresponding to each element of the optimal algebra, are obtained.

$\mathbf{X}_1$ gives a constant function, i.e. $u = c$, which is a trivial solution.

The invariants for the generator $\mathbf{X}_2$ are

$$F = ux^2 \text{ and } r = \frac{x^3}{t}. \quad (15)$$

The invariant equation is

$$(-r^2 + 24r)F'(r) + 3rF(r)F'(r) + 27r^3F'''(r)$$
$$-24F(r) - 2F^2(r) = 0. \quad (16)$$

The solution of this invariant equation is $F = r$.

Therefore, the solution of the KdV equation is $u(x,t) = \frac{x}{t}$.

For $\mathbf{X}_3, u(x,t) = \frac{x+k}{t}$.

For $\mathbf{X}_3$+$\mathbf{X}_4$, the invariants are $x - \frac{t^2}{2}$ and $F(r) = u - t$. The KdV equation takes the form $F''' + FF' + 1 = 0$, which is a third order nonlinear differential equation.

4. Symmetries of Nonlinear Filtration Equation

The motion of a non-Newtonian, weakly compressible fluid in a porous medium with a nonlinear filtration law

$$V = -\int D\left(\frac{\partial u}{\partial x}\right)\, d\left(\frac{\partial u}{\partial x}\right), \quad (17)$$

where V is the speed of filtration and u is the pressure is described by nonlinear filtration (NLF) equations [10,11]

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2}. \quad (18)$$

The function $D(\partial u/\partial x)$ is known as filtration coefficient. In general the filtration coefficient is not fixed. NLF equations have been solved for various filtration coefficients [10,11].

We choose

$$D\left(\frac{\partial u}{\partial x}\right) = \frac{1}{1+\left(\frac{\partial u}{\partial x}\right)^2}, \quad (19)$$

to find the discrete symmetries and group invariant solutions of this NLF equation which then lead to solutions under transformations due to the discrete symmetries.

The infinitesimal generators of (one-parameter) Lie groups of point symmetries of the NLF equation are [12].

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \mathbf{X}_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \quad (20)$$

$$\mathbf{X}_2 = \frac{\partial}{\partial t}, \mathbf{X}_5 = u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}. \quad (21)$$

$$\mathbf{X}_3 = \frac{\partial}{\partial u}, \quad (22)$$

Corresponding (one-parameter) Lie groups of point symmetries are

$$G_1\ (\hat{x}, \hat{t}, \hat{u}) = (x + \epsilon, t, u), \quad (23)$$

$$G_2\ (\hat{x}, \hat{t}, \hat{u}) = (x, t + \epsilon, u), \quad (24)$$

$$G_3\ (\hat{x}, \hat{t}, \hat{u}) = (x, t, u + \epsilon), \quad (25)$$

$$G_4\ (\hat{x}, \hat{t}, \hat{u}) = (e^{\epsilon}x, e^{2\epsilon}t, e^{\epsilon}u), \quad (26)$$

$$G_5\ (\hat{x}, \hat{t}, \hat{u}) = (x\cos\epsilon + u\sin\epsilon, t, -x\sin\epsilon + u\cos\epsilon). \quad (27)$$

Using these symmetries in the technique mentioned in Section 1, following discrete symmetries of the nonlinear filtration equation are obtained

$$\Gamma_{D1}\ (\hat{x}, \hat{t}, \hat{u}) = (\,-x, t, -u\,), \quad (28)$$

$$\Gamma_{D2}\ (\hat{x}, \hat{t}, \hat{u}) = (\,x, t, -u\,), \quad (29)$$

$$\Gamma_{D3}\ (\hat{x}, \hat{t}, \hat{u}) = (\,-x, t, u\,). \quad (30)$$

5. Exact Solutions

In this section an attempt has been made to find the group invariant solutions of the NLF equation due to the groups generated by $\mathbf{X}_i$, the basis generators. The group invariant solutions are then transformed using the discrete symmetries to obtain solutions.

For $\mathbf{X}_1$, $u(x,t) = F(t)$, where $F(t)$ is invariant. On substituting it in NLF equation we have $F(t) = c$, i.e. $u(x,t) = c$, which is trivial solution.

The symmetry generator $\mathbf{X}_2$, yields $u(x,t) = F(x)$. We now substitute this solution in NLF equation to determine F. We obtain

$$F(x) = c_1 x + c_2 = u(x,t). \tag{31}$$

There does not exist any group invariant solutions due to $\mathbf{X}_3$.

For $\mathbf{X}_4$, the invariants are $\frac{x}{\sqrt{t}} = r$ and $F = \frac{u}{\sqrt{t}}$. Writing NLF equation interms of these invariants to determine F.

$$2F''(r) = F(r) - rF'(r) + (F'(r))^2 F(r) - r(F'(r))^3. \tag{32}$$

This is a second order nonlinear ODE for F, whose solution is $F = r$. Hence, $u = x$ is the solution of the equation under consideration.

For $\mathbf{X}_5$, we have $v = u^2 + x^2$, and

$$t = r. \tag{33}$$

Therefore, we have

$$v = F(r), \tag{34}$$

We now substitute this solution in NLF equation to obtain

$$F(r) = -2r + c. \tag{35}$$

So

$$u(x,t) = \pm\sqrt{c_1 - 2t - x^2}, \tag{36}$$

All these solutions are due to the continuous symmetries of the NLF equation. Using discrete symmetries one more solution $u(x,t) = -x$ is also obtained.

6. Conclusions

There is a lot of literature available on the applications of continuous symmetries, in particular, on finding solutions of ordinary and partial differential equations. In recent past, Peter E. Hydon introduced a technique to obtain discrete symmetries using continuous symmetries. These discrete symmetries are then used to obtain some new solutions that could not be obtained by only using continuous symmetries. In this paper, using both discrete and continuous symmetries of the *Korteweg de Vries* (7) and the *nonlinear filtration equations*, group invariant solutions and the exact solutions of these equations are presented.

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